

Isoperimetric Hardy Type and Poincaré Inequalities on Metric Spaces

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Abstract We give a general construction of manifolds for which Hardy type operators characterize Poincaré inequalities. We also show a class of spaces where this property fails. As an application, we extend recent results of E. Milman to our setting.

1 Introduction

While working on sharp Sobolev–Poincaré inequalities in the classical Euclidean setting (cf. [17]) as well as the Gaussian setting (cf. [14]), we observed that the symmetrization methods we were developing could be readily extended to the more general setting of metric spaces (cf. [6, 14, 15, 16]). However, in the metric setting we found that we could not always decide if the results we had obtained were “sharp” or best possible.

Indeed, generally speaking, the methods that we use to show sharpness require the construction of special rearrangements and thus our spaces need to exhibit sufficient symmetries. In fact, in all the examples where we know how to prove sharpness, the “winning” rearrangements are those that are somehow connected with the solution of the underlying isoperimetric problems (for example, the symmetric decreasing rearrangements in the Euclidean case, which are associated with balls (cf. [17]), while in the Gaussian case one uses special rearrangements associated with half spaces (cf. [8, 14]) and, likewise,

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in the more general model cases of log concave measures (cf. [5, 7] and the more recent [1, 15, 16]). In particular, these special rearrangements allow us to show that there exist “special symmetrizations that do not increase the norm of the gradient,” i.e., that a suitable version of the Pólya–Szegő principle holds.

In preparation for a systematic study we observed that, in all the model cases we could treat, a key role was played by the boundedness of certain Hardy operators, which we termed “isoperimetric Hardy operators.” This led us to isolate the concept of “isoperimetric Hardy type spaces.” This property can be formulated in very general metric spaces and can be applied if we have estimates on the isoperimetric profiles. By formulating the problem in this fashion, while we may lose information about best constants, we gain the possibility of obtaining positive results that would be hard to obtain by other methods.

In this note, we continue this program and we address the question: which metric spaces are of “Hardy isoperimetric type”? On the positive side we show, using ideas of Ros [26], how to construct a class of metric spaces of “Hardy isoperimetric type” that contains all the model cases mentioned above. Therefore this construction provides us with a large class of spaces where our inequalities are sharp.

As another application we continue the discussion of the connection between our results and the recent work of E. Milman [23, 22, 24]), who has shown the equivalence, under convexity assumptions, of certain estimates for isoperimetric profiles. In [16], we extended and simplified E. Milman’s results to the setting of metric spaces of isoperimetric Hardy type. The construction presented in this paper thus gives a general concrete class of model spaces where Milman’s equivalences hold.

Finally on the negative side we also construct spaces that do not satisfy the “isoperimetric Hardy type condition.”

In the spirit of this book, we now comment briefly on the influence of Maz’ya’s work in our development. Underlying the equivalences of Theorem 2.1 below are two deep insights due to Maz’ya: Maz’ya’s fundamental result showing the equivalence between the Gagliardo–Nirenberg inequality and the isoperimetric inequality (cf. [18] and also [9]), and Maz’ya’s technique of showing self improvement of Sobolev’s inequality via smooth cut-offs (cf. [20]). Indeed, one of the themes of Theorem 2.1 is to develop the explicit connection of these two ideas using pointwise symmetrization inequalities (“symmetrization by truncation” cf. [17, 14]). Another theme of our method is that we formulate our inequalities incorporating directly geometric information, an idea that one can also already find in Maz’ya’s fundamental work characterizing Sobolev inequalities in rough domains [20] as well as in Maz’ya’s method characterizing Sobolev–Poincaré inequalities via isocapacity inequalities¹ (cf. [21]).

¹ A detailed discussion of the connection between Maz’ya’s isocapacity inequalities and symmetrization inequalities is given in [16].

As this outline shows, and we hope the rest of the paper proves, our methods owe a great deal to the pioneering work of Professor Maz'ya and we are grateful and honored by the opportunity to contribute to this book.

2 Background

Let (Ω, d, μ) be a metric probability space equipped with a separable Borel probability measure μ . Let $A \subset \Omega$ be a Borel set. Then the boundary measure or *Minkowski content* of A is by definition

$$\mu^+(A) = \liminf_{h \rightarrow 0} \frac{\mu(A_h) - \mu(A)}{h},$$

where $A_h = \{x \in \Omega : d(x, y) < h\}$ denotes the h -neighborhood of A .

The *isoperimetric profile* $I_{(\Omega, d, \mu)}$ is defined as the pointwise maximal function $I_{(\Omega, d, \mu)} : [0, 1] \rightarrow [0, \infty)$ such that

$$\mu^+(A) \geq I_{(\Omega, d, \mu)}(\mu(A))$$

holds for all Borel sets A .

Condition. We assume throughout that our metric spaces have isoperimetric profile functions $I_{(\Omega, d, \mu)}$ which are: continuous, concave, increasing on $(0, 1/2)$, symmetric about the point $1/2$, and vanish at zero².

A continuous function $I : [0, 1] \rightarrow [0, \infty)$, with $I(0) = 0$, concave, increasing on $(0, 1/2)$ and symmetric about the point $1/2$, and such that

$$I \geq I_{(\Omega, d, \mu)},$$

is called an *isoperimetric estimator* on (Ω, d, μ) .

For measurable functions $u : \Omega \rightarrow \mathbb{R}$ the distribution function of u is given by

$$\lambda_u(t) = \mu\{x \in \Omega : |u(x)| > t\} \quad (t > 0).$$

The *decreasing rearrangement* u^* of u is defined, as usual, by

$$u_\mu^*(s) = \inf\{t \geq 0 : \lambda_u(t) \leq s\} \quad (t \in (0, \mu(\Omega)]),$$

and we set

$$u_\mu^{**}(t) = \frac{1}{t} \int_0^t u_\mu^*(s) ds.$$

Given a locally Lipschitz real function, f defined on (Ω, d) (we write in what follows $f \in \text{Lip}(\Omega)$), the *modulus of the gradient* of f is defined by

² For a large class of examples, where these assumptions are satisfied we refer to [6, 23] and the references therein.

$$|\nabla f(x)| = \limsup_{d(x,y) \rightarrow 0} \frac{|f(x) - f(y)|}{d(x,y)}$$

and zero at isolated points³.

A Banach function space $X = X(\Omega)$ on (Ω, d, μ) is called a *rearrangement-invariant* (r.i.) space if $g \in X$ implies that all μ -measurable functions f with the same rearrangement function with respect to the measure μ , i.e., such that $f_\mu^* = g_\mu^*$, also belong to X ; moreover, $\|f\|_X = \|g\|_X$. An r.i. space $X(\Omega)$ can be represented by an r.i. space $\bar{X} = \bar{X}(0, 1)$ on the interval $(0, 1)$, with Lebesgue measure, such that

$$\|f\|_X = \|f_\mu^*\|_{\bar{X}},$$

for every $f \in X$. Typical examples of r.i. spaces are the L^p -spaces, Lorentz spaces, and Orlicz spaces. For more information we refer to [4].

In our recent work on symmetrization of Sobolev inequalities, we showed the following general theorem (cf. [15, 16] and the references therein)

Theorem 2.1. *Let $I : [0, 1] \rightarrow [0, \infty)$ be an isoperimetric estimator on (Ω, d, μ) . The following statements hold and are in fact equivalent.*

1. *Isoperimetric inequality*

$$\forall A \subset \Omega, \text{ Borel set, } \mu^+(A) \geq I(\mu(A)).$$

2. *Ledoux's inequality*

$$\forall f \in \text{Lip}(\Omega), \int_0^\infty I(\lambda_f(s)) ds \leq \int_\Omega |\nabla f(x)| d\mu(x).$$

3. *Maz'ya's inequality⁴*

$$\forall f \in \text{Lip}(\Omega), (-f_\mu^*)'(s)I(s) \leq \frac{d}{ds} \int_{\{|f| > f_\mu^*(s)\}} |\nabla f(x)| d\mu(x).$$

4. *Pólya-Szegő's inequality*

$$\forall f \in \text{Lip}(\Omega), \int_0^t ((-f_\mu^*)'(\cdot)I(\cdot))^*(s) ds \leq \int_0^t |\nabla f|_\mu^*(s) ds.$$

(The second rearrangement on the left hand side is with respect to the Lebesgue measure).

³ In fact, it is enough in order to define $|\nabla f|$ that f will be Lipschitz on every ball in (Ω, d) cf. [6, pp. 184,189] for more details.

⁴ See [19]; one can also find this inequality in [27, 28].

6. *Oscillation inequality*

$$\forall f \in \text{Lip}(\Omega), \quad (f_\mu^{**}(t) - f_\mu^*(t)) \leq \frac{t}{I(t)} |\nabla f|_\mu^{**}(t). \quad (2.1)$$

Given any rearrangement invariant space $X(\Omega)$, it follows readily from (2.1) that for all $f \in \text{Lip}(\Omega)$

$$\|f\|_{LS(X)} := \left\| (f_\mu^{**}(t) - f_\mu^*(t)) \frac{I(t)}{t} \right\|_{\overline{X}} \preceq \|\nabla f\|_X.$$

One salient characteristic of these spaces is that they explicitly incorporate in their definition the isoperimetric profiles associated with the geometry in question and thus they can automatically select the correct optimal spaces for different geometries (cf. [14, 15, 16] for more on this). While the $LS(X)$ spaces are not necessarily normed, often they are equivalent to normed spaces (cf. [25]), and, in the classical cases, lead to optimal Sobolev–Poincaré inequalities and embeddings (cf. [17, 13, 14] as well as [3, 2, 29] and the references therein).

3 Hardy Isoperimetric Type

Let Q_I be the operator defined on measurable functions on $(0, 1)$ by

$$Q_I f(t) = \int_t^1 f(s) \frac{ds}{I(s)},$$

where I is an isoperimetric estimator. We consider the possibility of completely characterizing Poincaré inequalities in terms of the boundedness of Q_I as an operator from \overline{X} to \overline{Y} .

In order to motivate what follows, we recall the following result obtained in [15, 16] for classical settings (cf. [14]).

Theorem 3.1. *Let X, Y be two r.i. spaces on Ω . Suppose that there exists a constant $c = c(X, Y)$ such that for every positive function $f \in \overline{X}$ with $\text{supp } f \subset (0, 1/2)$*

$$\|Q_I f(t)\|_{\overline{Y}} \leq c \|f_\mu^*\|_{\overline{X}}.$$

Then for all $g \in \text{Lip}(\Omega)$ ⁵

$$\left\| g - \int_\Omega g d\mu \right\|_Y \preceq \|\nabla g\|_X. \quad (3.1)$$

⁵ We note for future use that Poincaré inequalities can be equivalently formulated replacing $\int_\Omega g d\mu$ by a median value m of g , i.e., $\mu(g \geq m) \geq 1/2$ and $\mu(g \leq m) \geq 1/2$.

Furthermore, if the space \overline{X} is such that $\|f_\mu^*\|_{\overline{X}} \simeq \|f_\mu^{**}\|_{\overline{X}}$, then

$$\|f\|_Y \preceq \|f\|_{LS(X)} + \|f\|_{L^1}.$$

In fact, $LS(X)$ is an optimal space in the sense that if (3.1) holds, then for all $g \in \text{Lip}(\Omega)$

$$\left\| g - \int_\Omega g d\mu \right\|_Y \preceq \left\| g - \int_\Omega g d\mu \right\|_{LS(X)} \preceq \|\nabla g\|_X.$$

We give a simple, but nontrivial example that illustrates how the preceding developments allow us to transplant Sobolev–Poincaré inequalities to the metric setting.

Example. Suppose that (Ω, μ) has an isoperimetric estimator

$$I(s) \simeq s^{1-1/n} \quad (0 < s < 1/2).$$

It follows that, on functions supported on $(0, 1/2)$,

$$Q_I f(t) \simeq \int_t^{1/2} s^{1/n} f(s) \frac{ds}{s}.$$

Since the conditions for the boundedness Q_I on r.i. spaces are well understood, we can transplant the classical Sobolev inequalities to (Ω, μ) . Furthermore, we note that, in the borderline case $q = n$, the corresponding result using the optimal $LS(L^n)$ spaces is sharper than the classical Sobolev theorems (cf. [2]).

As was mentioned, it is known that the converse to Theorem 3.1 is true in a number of important classical cases. In other words, the operator Q_I in those cases gives a complete characterization of the Poincaré inequalities (for the most recent results cf. [15, 16]).

This led us to introduce the following definition.

Definition. We say that a probability metric space (Ω, d, μ) is of *isoperimetric Hardy type* if for any given isoperimetric estimator I , the following assertions are equivalent for all r.i. spaces $X = X(\Omega)$, $Y = Y(\Omega)$.

1. There exists $c = c(X, Y)$ such that

$$\forall f \in \text{Lip}(\Omega), \quad \left\| f - \int_\Omega f d\mu \right\|_Y \leq c \|\nabla f\|_X. \tag{3.2}$$

2. There exists $c = c(X, Y)$ such that

$$\|Q_I f\|_{\overline{Y}} \preceq \|f\|_{\overline{X}}, \quad f \in \overline{X} \text{ with } \text{supp}(f) \subset (0, 1/2).$$

4 Model Riemannian Manifolds

In this section, we construct a class of spaces of isoperimetric Hardy type spaces that includes the n -sphere \mathbb{S}^n (\mathbb{R}^n , γ_n) (\mathbb{R}^n with Gaussian measure) and symmetric log-concave probability measures on \mathbb{R} .

We follow the construction of Ros (cf. [26]). Let M_0 be a complete smooth oriented n_0 -dimensional Riemannian manifold with distance d . An absolutely continuous probability measure μ_0 w.r. to dV in M_0 is called a *model measure* if there exists a continuous family (in the sense of the Hausdorff distance on compact subsets) $\mathcal{D} = \{D^t : 0 \leq t \leq 1\}$ of closed subsets of M_0 satisfying the following conditions.

1. $\mu_0(D^t) = t$ and $D^s \subset D^t$, for $0 \leq s < t \leq 1$.
2. D^t is a smooth isoperimetric domain of μ_0 and $I_{\mu_0}(t) = \mu_0^+(D^t)$ is positive and smooth for $0 < t < 1$, where I_{μ_0} denotes the isoperimetric profile of M_0 .
3. The r -enlargement of D^t , defined by $(D^t)_r = \{x \in M_0 : d(x, D^t) \leq r\}$ verifies $(D^t)_r = D^s$ for some $s = s(t, r)$, $0 \leq t \leq 1$.
4. $D^1 = M_0$ and D^0 is either a point or the empty set.

Theorem 4.1. *Let (M_0, d) be an n_0 -dimensional Riemannian manifold endowed with a model measure μ_0 . Then (M_0, d) is of isoperimetric Hardy type.*

Proof. Consider the function defined by

$$\begin{aligned} p : M_0 &\rightarrow [0, 1], \\ x \in \partial D^t &\rightarrow t. \end{aligned}$$

Let $x, y \in M_0$ be such that $0 < p(y) < p(x)$. Let $D \in \mathcal{D}$ be such that $y \notin D$. Consider the function $h(r) = \mu_0(D_r)$, which is continuous and smooth for $0 < h(r) < 1$ and, in this range (cf. [26]),

$$h'(r) = I_{\mu_0}(h(r)). \quad (4.1)$$

From the definition of p it follows that $p(x) = h(d(x, D))$ and $p(y) = h(d(y, D))$. Since $d(x, D) - d(y, D) \leq d(x, y)$, we see that

$$\frac{p(x) - p(y)}{d(x, y)} \leq \frac{h(d(x, D)) - h(d(y, D))}{d(x, D) - d(y, D)} \leq \sup_s h'(s),$$

i.e., $p \in \text{Lip}(M_0)$ and

$$|\nabla p(x)| = \limsup_{y \rightarrow x} \left| \frac{p(x) - p(y)}{d(x, y)} \right|$$

is finite, it follows that $|\nabla p(x)|$ exists a.e. w.r. to dV (cf. [6, p. 2]) and hence a.e. w.r. μ_0 . Let us now compute $|\nabla p|$. Given $x \in M_0$ such that $p(x) = t < 1$, let $D \in \mathcal{D}$ so that $x \notin D$ and, as above, consider the function $h(r) = \mu_0(D_r)$. Let $z(x) \in M_0$ be such that

$$d(x, D) = d(x, z(x)).$$

Select y_n on the geodesic joining $z(x)$ and x such that $y_n \rightarrow x$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{p(x) - p(y_n)}{d(x, y_n)} \right| &= \lim_{n \rightarrow \infty} \left| \frac{h(d(x, D)) - h(d(y_n, D))}{d(x, D) - d(y_n, D)} \right| = h'(d(x, D)) \\ &= I_{\mu_0}(h(d(x, D))) \quad [\text{by (4.1)}] \\ &= I_{\mu_0}(p(x)). \end{aligned} \tag{4.2}$$

Let $f \in \overline{X}$ be a positive function with $\text{supp } f \subset (0, 1/2)$. Define

$$F(x) = \int_{p(x)}^1 f(s) \frac{ds}{I_{\mu_0}(s)}.$$

It is obvious that $F \in \text{Lip}(M_0)$ and, by (4.2),

$$|\nabla F(x)| = f(p(x)) \frac{1}{I_{\mu_0}(p(x))} |\nabla p(x)| = f(p(x)) \text{ a.e.}$$

We claim that the map $p : (M_0, \mu_0) \rightarrow ([0, 1], ds)$ is a measure-preserving transformation. To prove this claim, we need to see that for any measurable subset $R \subset [0, 1]$

$$\mu_0(p^{-1}(R)) = \int_R ds. \tag{4.3}$$

It is enough to see (4.3) for a closed interval. Let $[a, b] \subset [0, 1]$ ($0 \leq a < b \leq 1$). Then

$$\mu_0(p^{-1}([a, b])) = \mu_0(D^b) - \mu_0(D^a) = b - a.$$

Using this claim (cf. [4, Proposition 7.2, p. 80]) then a.e. we have

$$|F|_{\mu_0}^*(s) = \int_t^1 f(s) \frac{ds}{I_{\mu_0}(s)}, \quad |\nabla F|_{\mu_0}^*(s) = f^*(s).$$

It is obvious that the condition (3.2) is equivalent to

$$\|u - m\|_Y \preceq \|\nabla u\|_X,$$

where m is a median⁶ of f , now since $\mu_0(F = 0) \geq 1/2$, 0 is a median of F , and from

⁶ i.e., $\mu_0(f \geq m) \geq 1/2$ and $\mu_0(f \leq m) \geq 1/2$.

$$\|F - 0\|_Y \preceq \|\nabla F\|_X$$

we obtain

$$\left\| \int_t^1 f(s) \frac{ds}{I_{\mu_0}(s)} \right\|_{\overline{Y}} \preceq \|f\|_{\overline{X}}$$

as we wished to show. □

5 E. Milman’s Equivalence Theorems

In the next result, we have a list of progressively weaker statements that nevertheless have been shown by E. Milman (cf. [23, 22, 24]) to be equivalent under certain convexity assumptions. Likewise, E. Milman also has formulated similar results in the context of Orlicz spaces.

In [16], we have simplified and extended Milman’s results to the context of metric spaces with Hardy isoperimetric type, as well as considering general r.i. spaces.

Theorem 5.1. *Let (Ω, d, μ) be a space of Hardy isoperimetric type. Then the following statements are equivalent:*

(E1) *Cheeger’s inequality*

$$\exists C > 0 \text{ s.t. } I_{(\Omega, d, \mu)} \geq Ct, \quad t \in (0, 1/2].$$

(E2) *Poincaré inequality*

$$\exists P > 0 \text{ s.t. } \|f - m\|_{L^2(\Omega)} \leq P \|f\|_{L^2(\Omega)}.$$

(E3) *Exponential concentration: for all $f \in \text{Lip}(\Omega)$ with $\|f\|_{\text{Lip}(\Omega)} \leq 1$*

$$\exists c_1, c_2 > 0 \text{ s.t. } \mu\{|f - m| > t\} \leq c_1 e^{-c_2 t}, \quad t \in (0, 1).$$

(E4) *First moment inequality: for all $f \in \text{Lip}(\Omega)$ with $\|f\|_{\text{Lip}(\Omega)} \leq 1$*

$$\exists F > 0 \text{ s.t. } \|f - m\|_{L^1(\Omega)} \leq F.$$

Theorem 5.2. *Let (Ω, d, μ) be a space of isoperimetric Hardy type. Let $1 \leq q \leq \infty$, and let N be a Young function such that $\frac{N(t)^{1/q}}{t}$ is nondecreasing and there exists $\alpha > \max\{\frac{1}{q} - \frac{1}{2}, 0\}$ such that $\frac{N(t^\alpha)}{t}$ is nonincreasing. Then the following statements are equivalent.*

(E5) *(L_N, L^q) Poincaré inequality holds*

$$\exists P > 0 \text{ s.t. } \|f - m\|_{L_N(\Omega)} \leq P \|f\|_{L^q(\Omega)}.$$

(E6) Any isoperimetric profile estimator I satisfies: there exists a constant $c > 0$ such that

$$I(t) \geq c \frac{t^{1-1/q}}{N^{-1}(1/t)}, \quad t \in (0, 1/2].$$

The construction of the previous section thus provides a class of spaces where the previous theorems apply.

6 Some Spaces That Are not of Isoperimetric Hardy Type

In this section, we show that, unfortunately, not all metric spaces are of isoperimetric Hardy type.

Let $I : [0, 1] \rightarrow [0, \infty)$ be concave, continuous, increasing on $(0, 1/2)$, symmetric about the point $1/2$, and such that $I(0) = 0$. Let $0 \leq \beta \leq 1$. We say that I has β -asymptotic behavior if $\lim_{s \rightarrow 0^+} \frac{I(s)}{s^{1-\beta}}$ exists and lies on $(0, \infty)$.

Theorem 6.1. *Suppose that I is of β -asymptotic behavior. Then the following assertions hold.*

(i) *Given $0 < \beta < 1/2$, there is a metric space (Ω_0, d, μ) with $I(s) \simeq I_{(\Omega_0, d_0, \mu_0)}(s)$, a pair of r.i. spaces X, Y on Ω_0 , and a constant $c = c(X, Y)$ such that*

$$\left\| g - \int_{\Omega_0} g d\mu_0 \right\|_Y \leq c \|\nabla g\|_X, \quad g \in \text{Lip}(\Omega_0),$$

but $Q_I : \bar{X} \rightarrow \bar{Y}$ is not bounded.

(ii) *Given $0 < \beta < 1$, there is a metric space (Ω_1, d_1, μ_1) such that*

$$I(s) \simeq I_{(\Omega_1, d_1, \mu_1)}(s)$$

and (Ω_1, d, μ) is of isoperimetric Hardy type.

Proof. (i) (cf. [13] for a more general result). Let $1 < \alpha < 2$, and let Ω be an α -John domain on \mathbb{R}^2 ($|\Omega| = 1$). Then (cf. [11])

$$I_\Omega(s) \simeq s^{\alpha/2} = s^{1-(1-\alpha/2)}, \quad 0 \leq s \leq 1/2.$$

Let $t > 1$ be such that $\alpha > t - 1$, and let $r = \frac{2t}{\alpha+(1-t)}$. Note that $1 < t < r$. Then (cf. [12])

$$\left\| g - \int_{\Omega} g \right\|_{L^r} \preceq \|\nabla g\|_{L^t}.$$

In this case, the operator Q_{I_Ω} is given by

$$Q_{I_\Omega} f(t) = \int_t^1 u^{-\alpha/2} f(u) du.$$

Note that Q_{I_Ω} is not bounded from L^t to L^r . Indeed, the boundedness of Q_{I_Ω} can be reformulated as a weighted norm inequality for the operator $g \rightarrow \int_x^1 g(u) du$; namely,

$$\left\| \int_x^1 g(u) du \right\|_{L^r} \leq c \left\| g(x) x^{\alpha/2} \right\|_{L^t}. \tag{6.1}$$

It is well known that (6.1) holds if and only if (cf. [20])

$$\sup_{a>0} \left(\int_0^a 1 \right)^{1/r} \left(\int_a^1 \left(u^{\alpha t/2} \right)^{\frac{-1}{t-1}} du \right)^{\frac{t-1}{t}} < \infty. \tag{6.2}$$

Now, since $\alpha < 2$, it follows that $\frac{-\alpha t}{2(t-1)} + 1 < 0$, and for a near zero we have

$$\begin{aligned} \left(\int_0^a 1 \right)^{1/r} \left(\int_a^1 \left(u^{\alpha t/2} \right)^{\frac{-1}{t-1}} du \right)^{\frac{t-1}{t}} &\simeq a^{1/r} \left(a^{\frac{-\alpha t + 2(t-1)}{2(t-1)}} - 1 \right)^{\frac{t-1}{t}} \\ &\simeq a^{\frac{(1-t)(\alpha-1)}{2t}}. \end{aligned}$$

Consequently, since $\frac{(1-t)(\alpha-1)}{2t} < 0$, (6.2) cannot hold.

(ii) We follow Gallot’s method (cf. [10]) in order to construct (Ω_1, d_1, μ_1) . Let

$$B(r) = \int_r^1 \frac{ds}{I(s)}, \quad 0 \leq r \leq 1.$$

Since I is of β -asymptotic behavior, we see that $L = B(0) < \infty$. Since B is decreasing, it has an inverse which we denote by A . Consider the revolution surface $M = (0, L) \times \mathbb{S}^1$ (compactified by adjoining the points $\{0\} \times \mathbb{S}^1$ and $\{L\} \times \mathbb{S}^1$) provided with the Riemannian metric

$$g = dr^2 + I(A(r))^2 d\theta^2,$$

where $\theta \in \mathbb{S}^1$ and $d\theta^2$ is the canonical Riemannian metric on $(\mathbb{S}^1, \text{can})$. Notice that $I(A(0)) = I(A(L)) = 0$. We denote by Vol_M the volume of (M, g) . Multiplying the metric g by a constant, we can and will assume without loss that $\text{Vol}_M(M) = 1$. Denote by I_M the isoperimetric profile of (M, g, Vol_M) . Then (cf. [10, Appendix A.1.]) we can find a constant c , depending only on I , such that

$$cI(s) \leq I_M(s) \leq I(s).$$

‘ Let X, Y be two r.i. spaces on M such that

$$\left\| g - \int_M g d\text{Vol}_M \right\|_Y \preceq \|\nabla g\|_X, \quad g \in \text{Lip}(M).$$

Let f be a positive Lebesgue measurable function on $(0, 1)$ with $\text{supp} f \subset (0, 1/2)$. Define

$$u(r, \theta_1, \theta_2) = \int_{A(r)}^1 f(s) \frac{ds}{I(s)}, \quad (r, \theta_1, \theta_2) \in M.$$

It is plain that u is a Lipschitz function on M such that $\text{Vol}_M \{u = 0\} \geq 1/2$. Hence 0 is a median of u .

On the other hand, recall that (cf. [10, Page 57]) for any domain of revolution $\Omega(\lambda) = (0, \lambda) \times \mathbb{S}^1 \subset M$ we have that

$$\text{Vol}_M^+(\partial\Omega(\lambda)) = I(\text{Vol}_M(\Omega(\lambda))).$$

In other words,

$$A'(r) = I(A(r)).$$

Therefore,

$$|\nabla u(r, \theta_1, \theta_2)| = \left| \frac{\partial}{\partial r} u(r) \right| = \left| -f(A(r)) \frac{A'(r)}{I(A(r))} \right| = f(A(r)).$$

Now, since

$$u_{\text{Vol}_M}^*(t) = \int_t^1 f(s) \frac{ds}{I(s)}, \quad |\nabla u|_{\text{Vol}_M}^*(t) = f^*(t),$$

from

$$\|u - 0\|_Y \preceq \|\nabla u\|_X$$

we deduce that

$$\left\| \int_t^1 f(s) \frac{ds}{I(s)} \right\|_Y \preceq \|f\|_X,$$

as we wished to show. □

By Theorem 6.1 the verification of a Sobolev–Poincaré inequality cannot be reduced, in general, to establish the boundedness of the associated isoperimetric Hardy operator. However, if the profiles are of β -asymptotic behavior, we have the following weaker positive result.

Theorem 6.2. *Let I be of β -asymptotic behavior ($0 < \beta < 1$). Let \mathcal{M}_I be the set of metric probability spaces (Ω, d, μ) such that*

$$I_{(\Omega, d, \mu)} \geq I.$$

Let \bar{X}, \bar{Y} be two r.i. spaces on $[0, 1]$. Then the following statements are equivalent.

1.

$$\inf_{(\Omega, d, \mu) \in \mathcal{M}_h} \inf_{g \in \text{Lip}(\Omega)} \frac{\| |\nabla g|_\mu^* \|_{\bar{X}}}{\| (g - \int_\Omega g d\mu)_\mu^* \|_{\bar{Y}}} = c > 0.$$

2.

$$Q_I : \bar{X} \rightarrow \bar{Y} \text{ is bounded.} \quad (6.3)$$

Proof. 1 \rightarrow 2. Given I of β -asymptotic behavior, consider the revolution surface M constructed in part (ii) of the previous theorem. Since $M \in \mathcal{M}_I$, by hypothesis:

$$\left\| \left(g - \int_\Omega g d\mu \right)_\mu^* \right\|_{\bar{Y}} \leq c \| (\nabla g)_\mu^* \|_{\bar{X}}, \quad g \in \text{Lip}(M),$$

which is equivalent to (6.3) since M is of Hardy isoperimetric type.

2 \rightarrow 1. The implication is a direct consequence of Theorem 3.1. \square

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