LOCAL AND GLOBAL ESTIMATES FOR HYPERBOLIC EQUATIONS IN BESOV-LIPSCHITZ AND TRIEBEL-LIZORKIN SPACES

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Abstract. In this paper we establish optimal local and global Besov-Lipschitz and Triebel-Lizorkin estimates for the solutions to linear hyperbolic partial differential equations. These estimates are based on local and global estimates for Fourier integral operators that span all possible scales (and in particular both Banach and quasi-Banach scales) of Besov-Lipschitz spaces $B^s_{p,q}(\mathbb{R}^n)$, and certain Banach and quasi-Banach scales of Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$.

1. Introduction

Consider the following Cauchy problem for the wave equation in $\mathbb{R}^{n+1}$,

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = 0, \quad t \neq 0,\, x \in \mathbb{R}^n, \\ u(0,x) = f_0(x), \\ \partial_t u(0,x) = f_1(x). \end{cases}$$

In [1] P. Brenner showed that for a fixed time $\tau > 0$ the solution to this problem verifies the estimate

$$\|u(\tau,\cdot)\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant C_\tau \left(\|f_0\|_{B^{s+\nu}_{p',q}(\mathbb{R}^n)} + \|f_1\|_{B^{s+\nu-1}_{p',q}(\mathbb{R}^n)} \right), \tag{1}$$

where $s \in \mathbb{R}$, $p \in [2,\infty)$, $p' = \dfrac{p}{p-1}$, $q \in [1,\infty]$, and $(n+1)\left|\dfrac{1}{p} - \dfrac{1}{2}\right| \leqslant \nu \leqslant 2n \left|\dfrac{1}{p} - \dfrac{1}{2}\right|$.

In [7] L. V. Kapitanskiĭ, extended and improved the results of Brenner to the range $p \in [2,\infty]$ and $(n-1)\left|\dfrac{1}{p} - \dfrac{1}{2}\right| \leqslant \nu \leqslant n \left|1 - \dfrac{2}{p}\right|$. In fact Kapitanskiĭ's result also applies to more general variable coefficient second order strictly hyperbolic equations, and also is valid in the realm of Triebel-Lizorkin spaces for the same range of parameters.

Later, J. Ginibre and G. Velo [3] established Strichartz-type estimates for homogeneous Besov-Lipschitz and Triebel-Lizorkin spaces which are useful in the applications to non-linear hyperbolic problems.

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However, the pioneering results of Brenner's were achieved by establishing $L^p \to L^q$ estimates for a class of Fourier integral operators that appear naturally in the construction of solutions (or parametrises) for strictly hyperbolic partial differential equations.

The next breakthrough was made in [20], where A. Seeger, C. Sogge and E. Stein showed that for every smooth spatial cut-off function χ one has the estimate

$$\|\chi\, u(\tau,\cdot)\|_{H^{s,p}(\mathbb{R}^n)} \leqslant C_\tau \left(\|f_0\|_{H^{s+\nu,p}(\mathbb{R}^n)} + \|f_1\|_{H^{s+\nu-1,p}(\mathbb{R}^n)}\right), \tag{2}$$

for $s \in \mathbb{R}$, $\nu = (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$ and $p \in (1,\infty)$.

As a consequence of this, one has

$$\|\chi\, u(\tau,\cdot)\|_{B^s_{2,2}(\mathbb{R}^n)} \leqslant C_\tau \left(\|f_0\|_{B^s_{2,2}(\mathbb{R}^n)} + \|f_1\|_{B^{s-1}_{2,2}(\mathbb{R}^n)}\right),$$

with $s \in \mathbb{R}$. Moreover, in [20] it was also proven that

$$\|\chi\, u(\tau,\cdot)\|_{B^s_{\infty,\infty}(\mathbb{R}^n)} \leqslant C_\tau \left(\|f_0\|_{B^{s+\nu}_{\infty,\infty}(\mathbb{R}^n)} + \|f_1\|_{B^{s+\nu-1}_{\infty,\infty}(\mathbb{R}^n)}\right),$$

for $\nu = \dfrac{n-1}{2}$. This is of course nothing but the Lipschitz space estimate in [20].

In this paper we extend (1) to the optimal global estimate

$$\|u(\tau,\cdot)\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant C_\tau \left(\|f_0\|_{B^{s+\nu}_{p,q}(\mathbb{R}^n)} + \|f_1\|_{B^{s+\nu-1}_{p,q}(\mathbb{R}^n)}\right),$$

where $s \in \mathbb{R}$, $p \in \left(\frac{n}{n+1}, \infty\right]$, $q \in (0,\infty]$, and $\nu = (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$.
Moreover we also show that the local version of the above estimate is valid for $s \in \mathbb{R}$, $p \in (0,\infty]$ and $q \in (0,\infty)$. Furthermore we show the following global estimate for the Triebel-Lizorkin spaces

$$\|u(\tau,\cdot)\|_{F^s_{p,q}(\mathbb{R}^n)} \leqslant C_\tau \left(\|f_0\|_{F^{s+\nu}_{p,q}(\mathbb{R}^n)} + \|f_1\|_{F^{s+\nu-1}_{p,q}(\mathbb{R}^n)}\right),$$

where $s \in \mathbb{R}$, $p \in \left(\frac{n}{n+1}, \infty\right]$, $\min(2,p) \leqslant q \leqslant \max(2,p)$, and $\nu = (n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$.
At the local level, we can improve the range of p in the above estimate to $(0,\infty)$. However if one assumes that $\nu < -(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$, then the range of the Triebel-Lizorkin estimate above is improved to the optimal range $p, q \in (0,\infty]$ in the local case, and $p \in \left(\frac{n}{n+1}, \infty\right]$, $q \in (0,\infty]$ in the global case. Moreover, as was done in [7] and [20], we also establish similar estimates for more general variable coefficient hyperbolic PDEs.

All of these results are achieved through proving sharp local and global estimates for Fourier integral operators of the form

$$T^\varphi_a f(x) = \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\varphi(x,\xi)}\, \widehat{f}(\xi)\, đ\xi$$

on Besov-Lipschitz and Triebel-Lizorkin spaces. The interest in these spaces stems from the fact that they contain spaces such as Lebesgue spaces, Lipschitz spaces (Hölder spaces), Sobolev spaces, Hardy spaces and BMO spaces, as special cases. Moreover these spaces also contain scales that are quasi-Banach and indeed one of the purposes of the this paper is to extend the estimates for the solutions of the wave equation to the quasi-Banach setting. It turns out that in the context of global estimates for Fourier integral operators, the restriction for p being in $\left(\frac{n}{n+1}, \infty\right]$, is sharp for the validity of global estimates, since we can produce counter-examples to the global boundedness of the Fourier integral operators for $p \in \left(0, \frac{n}{n+1}\right]$. However, if one is looking for local estimates, as in for example [20], then we show that in that case the range of the p's can indeed be improved to the full range $(0, \infty]$. We should also mention that although optimal local L^p estimates for Fourier integral operators are by now classical (see [20]), the optimal global L^p estimates for these operators are rather recent (see [18]).

The paper is organised as follows; in Section 2 we recall some definitions, facts and results from microlocal and harmonic analysis that will be used throughout the paper. In Section 3 we decompose the Fourier integral operators into certain pieces and establish the basic kernel estimates for these pieces. The kernel estimates are used in the proof of the regularity in both Besov-Lipschitz and Triebel-Lizorkin spaces in the later sections. In Section 4 we describe the transference of local to global regularity of Fourier integral operators due to M. Ruzhansky and M. Sugimoto, and how it can be fit into our setting. In Section 5 we prove the optimal local and global boundedness of Fourier integral operators on all possible scales of Besov-Lipschitz spaces (Theorem 5.7). In Section 6 we deal with the regularity problem in certain scales of Triebel-Lizorkin spaces and obtain optimal results for those scales (Theorem 7.1). However, we also show that if the order of the operator is just below the critical threshold, then the Triebel-Lizorkin regularity can be extended to all possible scales of the Triebel-Lizorkin spaces (Theorem 6.1). In Section 8 we prove the optimal one dimensional results regarding the regularity of Fourier integral operators for all possible Banach and quasi-Banach scales. In Section 9 we give a motivation for why the boundedness results that we have obtained are sharp, and finally in Section 10 we produce the aforementioned local and global Besov-Lipschitz and Triebel-Lizorkin space estimates for hyperbolic partial differential equations (estimates (58) and (59) and Theorem 10.1).

As is common practice, we will denote constants which can be determined by known parameters in a given situation, but whose value is not crucial to the problem at hand, by C. Such parameters in this paper would be, for example, m, p, q, s, n, and the constants connected to the seminorms of various amplitudes or phase functions. The value of C may differ from line to line, but in each instance could be estimated if necessary. We also write $a \lesssim b$ as shorthand for $a \leqslant Cb$.

2. Definitions and Preliminaries

In this section, we will collect all the definitions that will be used throughout this paper. We also state some useful results from both harmonic and microlocal analysis which will be used in the proofs of our results.

Let us recall the definition of the standard *Littlewood-Paley decomposition* which is a basic ingredient in our proofs, and is also used to define the function spaces that we are concerned with here.

Definition 2.1. *Let* $\psi_0 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ *be equal to* 1 *on* $B(0,1)$ *and have its support in* $B(0,2)$*. Then let*

$$\psi_j(\xi) := \psi_0\left(2^{-j}\xi\right) - \psi_0\left(2^{-(j-1)}\xi\right),$$

where $j \geqslant 1$ *is an integer and* $\psi(\xi) := \psi_1(\xi)$*. Then* $\psi_j(\xi) = \psi\left(2^{-(j-1)}\xi\right)$ *and one has the following Littlewood-Paley partition of unity*

$$\sum_{j=0}^{\infty} \psi_j(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^n.$$

It is sometimes also useful to define a sequence of smooth and compactly supported functions Ψ_j *with* $\Psi_j = 1$ *on the support of* ψ_j *and* $\Psi_j = 0$ *outside a slightly larger compact set. Explicitly, one could set*

$$\Psi_j := \psi_{j+1} + \psi_j + \psi_{j-1},$$

with $\psi_{-1} := 0$.

Using the Littlewood-Paley decomposition of Definition 2.1, one can define the so called *Besov-Lipschitz spaces* which are one the main function spaces from the point of view of this paper.

Definition 2.2. *Let* $0 < p, q \leq \infty$ *and* $s \in \mathbb{R}$*. The Besov-Lipschitz spaces are defined by*

$$B^s_{p,q}(\mathbb{R}^n) := \left\{ f \in \mathscr{S}'(\mathbb{R}^n) \,:\, \|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} 2^{jqs} \|\psi_j(D)f\|^q_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{q}} < \infty \right\}.$$

It is also worth to mention that for $p = q = \infty$ *and* $0 < s \leqslant 1$ *we obtain the familiar Lipschitz space* $\Lambda^s(\mathbb{R}^n)$*, i.e.* $B^s_{\infty,\infty}(\mathbb{R}^n) = \Lambda^s(\mathbb{R}^n)$.

Remark 2.3. *Different choices of the basis* $(\psi_j)_{j=0}^\infty$ *give equivalent (quasi)-norms of* $B^s_{p,q}(\mathbb{R}^n)$ *in* Definition 2.2*, see e.g.* [23]*. We will use either* $(\psi_j)_{j=0}^\infty$ *or* $(\Psi_j)_{j=0}^\infty$ *to define the norm of* $B^s_{p,q}(\mathbb{R}^n)$.

We will also produce boundedness results in the realm of *Triebel-Lizorkin* spaces which can be defined using Littlewood-Paley theory, as follows:

Definition 2.4. *Let* $0 < p, q \leq \infty$ *and* $s \in \mathbb{R}$*. The Triebel-Lizorkin spaces are defined by*

$$F^s_{p,q}(\mathbb{R}^n) := \left\{ f \in \mathscr{S}'(\mathbb{R}^n) \,:\, \|f\|_{F^s_{p,q}(\mathbb{R}^n)} := \left\| \left(\sum_{j=0}^{\infty} 2^{jqs} |\psi_j(D)f|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} < \infty \right\}.$$

Note that for $-\infty < s < \infty$ *and* $1 \leqslant p < \infty$, $F^s_{p,2}(\mathbb{R}^n) = H^{s,p}(\mathbb{R}^n)$ (*various* L^p*-based Sobolev and Sobolev-Slobodeckij spaces*) *and for* $0 < p < \infty$ *and* $F^0_{p,2}(\mathbb{R}^n) = h^p(\mathbb{R}^n)$ (*the local Hardy spaces*). *Moreover the dual space of* $F^0_{1,2}(\mathbb{R}^n)$ *is* bmo (*the local* BMO).

Another fact which will be useful to us is that for $-\infty < s < \infty$ and $0 < p \leqslant \infty$

$$B^s_{p.p}(\mathbb{R}^n) = F^s_{p,p}(\mathbb{R}^n), \tag{3}$$

and one has the continuous embedding

$$F^{s+\varepsilon}_{p,q_0}(\mathbb{R}^n) \hookrightarrow F^s_{p,q_1}(\mathbb{R}^n), \tag{4}$$

for $-\infty < s < \infty$ and $0 < p < \infty$ and $0 < q_0 \leqslant \infty$, $0 < q_1 \leqslant \infty$ and all $\varepsilon > 0$. Furthermore, for $s' \in \mathbb{R}$, the operator $(1-\Delta)^{\frac{s'}{2}}$ maps $F^s_{p,q}(\mathbb{R}^n)$ isomorphically into $F^{s-s'}_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ isomorphically into $B^{s-s'}_{p,q}(\mathbb{R}^n)$.

Since we shall later on specifically deal with Triebel-Lizorkin spaces $F^0_{p,2}(\mathbb{R}^n) = h^p(\mathbb{R}^n)$, we also recall that a function a is called a h^p-atom if for some $x_0 \in \mathbb{R}^n$ and $r > 0$ the following three conditions are satisfied:

(i) $\operatorname{supp} a \subset B(x_0, r)$,

(ii) $|a(x)| \leqslant |B(x_0,r)|^{-\frac{1}{p}}$,

(iii) if $r \leqslant 1$ and $|\alpha| \leqslant M = 1 + \left[n\left(\frac{1}{p} - 1\right)\right]$ then $\int_{\mathbb{R}^n} x^\alpha a(x)\,\mathrm{d}x = 0$, and no further condition if $r > 1$. Here $[x]$ denotes the integer part of x.

It is well known (see [23]) that a distribution $f \in h^p(\mathbb{R}^n)$ has an atomic decomposition

$$f = \sum_j \lambda_j a_j,$$

where the λ_j are constants such that

$$\sum_j |\lambda_j|^p \approx \|f\|^p_{h^p(\mathbb{R}^n)} = \|f\|^p_{F^0_{p,2}(\mathbb{R}^n)}$$

and the a_j are h^p-atoms.

Another important and useful fact about Besov-Lipschitz and Triebel-Lizorkin spaces is the following:

Theorem 2.5. *Let* $\eta : \mathbb{R}^n \to \mathbb{R}^n$ *with* $\eta(x) = (\eta_1(x), \dots, \eta_n(x))$ *be a diffeomorphism such that* $|\det D\eta(x)| \geqslant c > 0$, $\forall x \in \mathbb{R}^n$ ($D\eta$ *denotes the Jacobian matrix of* η), *and* $\|\partial^\alpha \eta_j(x)\|_{L^\infty(\mathbb{R}^n)} \lesssim 1$ *for all* $j \in \{1, \dots, n\}$ *and* $|\alpha| \geqslant 1$. *Then for* $s \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < q \leqslant \infty$ *one has*

$$\|f \circ \eta\|_{F^s_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^s_{p,q}(\mathbb{R}^n)}.$$

The same invariance estimate is also true for Besov-Lipschitz spaces $B^s_{p,q}(\mathbb{R}^n)$ *for* $s \in \mathbb{R}$, $0 < p \leqslant \infty$ *and* $0 < q \leqslant \infty$.

For a proof see J. Johnsen, S. Munch Hansen and W. Sickel [5] Corollary 25, and H. Triebel [25] Theorem 4.3.2. References [23] and [25] and [26] are actually the standard references for all the facts concerning Besov-Lipschitz and Triebel-Lizorkin spaces. See also [24] for a summary of most important properties of the Triebel-Lizorkin spaces.

Next we recall the definition of two classes of amplitudes which are the basic building blocks of the pseudodifferential and the Fourier integral operators used in this paper. The first class was first introduced by J.J. Kohn and L. Nirenberg in [10].

Definition 2.6. *An amplitude (symbol) $a(x,\xi)$ in the class $S^m(\mathbb{R}^n)$ is a function $a \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ that verifies the estimate*

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \lesssim \langle\xi\rangle^{m-|\alpha|},$$

for all multi-indices α and β and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$. We shall henceforth refer to m as the order of the amplitude.

The second class of amplitudes that we shall use in this paper are those which have no regularity in the x variable, which were first introduced by C. Kenig and W. Staubach in [9].

Definition 2.7. *An amplitude (symbol) $a(x,\xi)$ is in the class $L^\infty S^m(\mathbb{R}^n)$ if it is essentially bounded in the x variable, $\mathcal{C}^\infty(\mathbb{R}^n)$ in the ξ variable and verifies the estimate*

$$\left\|\partial_\xi^\alpha a(\cdot,\xi)\right\|_{L^\infty(\mathbb{R}^n)} \lesssim \langle\xi\rangle^{m-|\alpha|},$$

for all multi-indices α and $\xi \in \mathbb{R}^n$. Later, N. Michalowski, D. Rule *and* W. Staubach *in* [11] *introduced the class of rough compound amplitudes $L^\infty A^m(\mathbb{R}^n)$ which consists of all $a(x,y,\xi)$ that verify the estimate*

$$\left\|\partial_\xi^\alpha a(\cdot,\cdot,\xi)\right\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} \lesssim \langle\xi\rangle^{m-|\alpha|},$$

for all multi-indices α and $\xi \in \mathbb{R}^n$.

We note that $S^m(\mathbb{R}^n) \subset L^\infty S^m(\mathbb{R}^n)$.

For the purpose of proving boundedness results for Fourier integral operators, it turns out that the following order of the amplitude is the critical one, namely

$$m_c(p) := -(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|, \tag{5}$$

where $0 < p \leqslant \infty$. This means that, we will be able to establish various boundedness results for the Fourier integral operators when the order of the amplitude is less than or equal to $m_c(p)$.

Given the symbol classes defined above, one associates to the symbol its *Kohn-Nirenberg quantisation* as follows:

Definition 2.8. *Let a be a symbol. Define a pseudodifferential operator* (ΨDO *for short*) *as the operator*

$$a(x,D)f(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\widehat{f}(\xi)\,đ\xi,$$

a priori defined on the Schwartz class $\mathscr{S}(\mathbb{R}^n)$. Here and in what follows, $đ\xi := (2\pi)^{-n}\,d\xi$.

In order the define the Fourier integral operators that are studied in this paper, we also define the classes of phase functions, which together with the amplitudes of Definitions 2.6 and 2.7 are the main building blocks of Fourier integral operators.

Definition 2.9. *A phase function* $\varphi(x,\xi)$ *in the class* Φ^k *is a function* $\varphi(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$, *positively homogeneous of degree* 1 *in the frequency variable* ξ *satisfying the following estimate*

$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}} |\xi|^{-1+|\alpha|} \left|\partial_\xi^\alpha \partial_x^\beta \varphi(x,\xi)\right| \leqslant C_{\alpha,\beta}, \tag{6}$$

for any pair of multi-indices α *and* β, *satisfying* $|\alpha|+|\beta| \geqslant k$. *In this paper we will mainly use phases in class* Φ^2 *and occasionally also* Φ^1.

We will also need to consider phase functions that satisfy certain *non-degeneracy conditions*. These conditions have to be adapted to the case of local and global boundedness in an appropriate way. Following [20], in connection to the investigation of the local results, that is, under the assumption that the x support of the amplitude $a(x,\xi)$ lies within a fixed compact set $\mathcal{K}$, the non-degeneracy condition is formulated as follows:

Definition 2.10. *Let* $\mathcal{K}$ *be a fixed compact subset of* $\mathbb{R}^n$. *One says that the phase function* $\varphi(x,\xi)$ *satisfies the non-degeneracy condition if*

$$\det\left(\partial^2_{x_j\xi_k}\varphi(x,\xi)\right) \neq 0, \qquad \text{for all } (x,\xi)\in\mathcal{K}\times\mathbb{R}^n\setminus\{0\}. \tag{7}$$

Following the approach in e.g. [17], for the global L^p boundedness results that were established in that paper, we also define the following somewhat stronger notion of non-degeneracy:

Definition 2.11. *One says that the phase function* $\varphi(x,\xi)$ *satisfies the strong non-degeneracy condition* (*or* φ *is* SND *for short*) *if*

$$\left|\det\left(\partial^2_{x_j\xi_k}\varphi(x,\xi)\right)\right| \geqslant \delta, \qquad \text{for some } \delta>0 \text{ and all } (x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}. \tag{8}$$

Having the definitions of the amplitudes and the phase functions at hand, one has

Definition 2.12. *A Fourier integral operator* (FIO *for short*) T_a^φ *with amplitude* a *and phase function* φ *satisfying* (7), *is defined* (*once again a-priori on* $\mathscr{S}(\mathbb{R}^n)$) *by*

$$T_a^\varphi f(x) := \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi)\widehat{f}(\xi)\,đ\xi. \tag{9}$$

The following composition result, whose proof can be found in [16, Theorem 4.2] will enable us to keep track of the parameter while a parameter-dependent ΨDO is composed with a parameter-dependent FIO. This will be crucial in some of the forthcoming proofs.

Theorem 2.13. *Let* $m \leqslant 0$, $0 < \varepsilon < \dfrac{1}{2}$ *and* $\Omega := \mathbb{R}^n \times \{|\xi| > 1\}$. *Suppose that* $a_t(x,\xi) \in S^m(\mathbb{R}^n)$ *uniformly in* $t \in (0,1]$ *and it is supported in* Ω, $\rho(\xi) \in S^0(\mathbb{R}^n)$ *and* $\varphi \in \mathcal{C}^\infty(\Omega)$ *is such that*

(i) *for constants* $C_1, C_2 > 0$, $C_1|\xi| \leqslant |\nabla_x \varphi(x,\xi)| \leqslant C_2|\xi|$ *for all* $(x,\xi) \in \Omega$, *and*
(ii) *for all* $|\alpha|, |\beta| \geqslant 1$, $|\partial_x^\alpha \varphi(x,\xi)| \lesssim \langle \xi \rangle$ *and* $|\partial_\xi^\alpha \partial_x^\beta \varphi(x,\xi)| \lesssim |\xi|^{1-|\alpha|}$, *for all* $(x,\xi) \in \Omega$.

Consider the parameter dependent Fourier multiplier and the Fourier integral operator

$$\rho(tD)f(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi} \rho(t\xi) \widehat{f}(\xi)\, \text{đ}\xi \text{ and } T^\varphi_{a_t}(f)(x) := \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a_t(x,\xi) \widehat{f}(\xi)\, \text{đ}\xi$$

and let σ_t *be the amplitude of the composition operator* $\rho(tD)T^\varphi_{a_t} = T^\varphi_{\sigma_t}$ *given by*

$$\sigma_t(x,\xi) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} a_t(y,\xi)\rho(t\eta) e^{i(x-y)\cdot\eta + i\varphi(y,\xi) - i\varphi(x,\xi)}\, \text{đ}\eta\, \mathrm{d}y.$$

Then, for each $M \geqslant 1$, *we can write* σ_t *as*

$$\sigma_t(x,\xi) = \rho(t\nabla_x\varphi(x,\xi)) a_t(x,\xi) + \sum_{0<|\alpha|<M} \frac{t^{|\alpha|}}{\alpha!} \sigma_\alpha(t,x,\xi) + t^{M\varepsilon} r(t,x,\xi)$$

for $t \in (0,1)$. *Moreover, for all multi-indices* β, γ *one has*

$$\sup_{t\in(0,1)} \left| \partial_\xi^\gamma \partial_x^\beta \sigma_\alpha(t,x,\xi) t^{|\alpha|(\varepsilon-1)} \right| \lesssim \langle \xi \rangle^{m-|\alpha|\left(\frac{1}{2}-\varepsilon\right)-|\gamma|} \text{ for } 0 < |\alpha| < M,$$

and

$$\sup_{t\in(0,1)} \left| \partial_\xi^\gamma \partial_x^\beta r(t,x,\xi) \right| \lesssim \langle \xi \rangle^{m-M\left(\frac{1}{2}-\varepsilon\right)-|\gamma|}.$$

To deal with the low frequency portion of the kernels of FIOs, which are frequency supported in a neighbourhood of the origin (where the phase function is singular), the following lemma which was proven in [2, Lemma 1.17], will come in handy.

Lemma 2.14. *Let* $b(x,\xi)$ *be a bounded function which belongs to* $\mathcal{C}_c^{n+1}(\mathbb{R}^n \setminus \{0\})$ *in the* ξ *variable. Moreover assume that* $b(x,\xi)$ *satisfies*

$$\sup_{\xi\in\mathbb{R}^n\setminus\{0\}} |\xi|^{-1+|\alpha|} \left\| \partial_\xi^\alpha b(\cdot,\xi) \right\|_{L^\infty(\mathbb{R}^n)} < \infty,$$

for $|\alpha| = n+1$. *Then for all* $\mu \in [0,1)$

$$\sup_{x,y\in\mathbb{R}^n} \langle y \rangle^{n+\mu} \left| \int_{\mathbb{R}^n} e^{-iy\cdot\xi}\, b(x,\xi)\, \text{đ}\xi \right| < \infty.$$

The following phase reduction lemma, whose proof can be found in [2, Lemma 1.10], will reduce the phase of the Fourier integral operators to a linear term plus a phase for which the first order frequency derivatives are bounded.

Lemma 2.15. *Any Fourier integral operator T_σ^φ of the type* (9) *with amplitude $\sigma(x,\xi) \in S^m(\mathbb{R}^n)$ and phase function $\varphi(x,\xi) \in \Phi^2$, can be written as a finite sum of operators of the form*

$$\int_{\mathbb{R}^n} a(x,\xi)\, e^{i\theta(x,\xi)+i\nabla_\xi\varphi(x,\zeta)\cdot\xi}\, \widehat{u}(\xi)\, \mathrm{đ}\xi$$

where ζ is a point on the unit sphere $\mathbb{S}^{n-1}$, $\theta(x,\xi) \in \Phi^1$, and $a(x,\xi)$ is localised in the ξ variable around the point ζ. Moreover, if one has a Fourier integral operator of the form

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} a(y,\xi)\, e^{i\varphi(y,\xi)-ix\cdot\xi}\, u(y)\, \mathrm{đ}\xi\, \mathrm{d}y,$$

with $\varphi \in \Phi^2$ then this operator can be written as a finite sum of operators

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} a(y,\xi)\, e^{i\theta(y,\xi)+i\nabla_\xi\varphi(y,\zeta)\cdot\xi-ix\cdot\xi}\, u(y)\, \mathrm{đ}\xi\, \mathrm{d}y,$$

where $\theta(y,\xi) \in \Phi^1$, and $a(y,\xi)$ is localised in the ξ variable around the point ζ.

We will state the following lemma originally due to J. Peetre [13], whose proof can be found in [23, Section 2.3.6], which in combination with the previous lemma, turns out to be very useful later on in proving the boundedness of the low frequency part of FIOs.

Lemma 2.16. *Let $f \in \mathcal{C}^1(\mathbb{R}^n)$ with Fourier support inside the unit ball. Then for every $r,\rho > 0$, with $r \geqslant \dfrac{n}{\rho}$ one has*

$$\big(\langle\cdot\rangle^{-\rho} * |f|\big)(x) \lesssim \Big(M(|f|^r)(x)\Big)^{\frac{1}{r}},$$

where M denotes the Hardy-Littlewood maximal function on $\mathbb{R}^n$.

Since pseudodifferential operators are not in general L^p bounded for $0 < p \leqslant 1$, we will also need a weaker version of an L^p space. Hence, following H. Triebel [23], we define the L^p spaces with compact Fourier support.

Definition 2.17. *Let $0 < p \leqslant \infty$ and $\mathcal{K} \subset \mathbb{R}^n$ be a compact set. Define*

$$L^p_{\mathcal{K}}(\mathbb{R}^n) := \Big\{ f \in \mathscr{S}'(\mathbb{R}^n) \,:\, \|f\|_{L^p(\mathbb{R}^n)} < \infty, \operatorname{supp}\widehat{f} \in \mathcal{K} \Big\}$$

Observe that other authors may use the notation $L_p^{\mathcal{K}}(\mathbb{R}^n)$, see e.g. [23].

In connection to this and the convolution of functions in $L^p_{\mathcal{K}}(\mathbb{R}^n)$ spaces, the following lemma, whose proof can be found in Remark 2 of [23, p. 28], is quite useful.

Lemma 2.18. *Let $\mathcal{K} := \overline{B(0,r)}$ for some $r > 0$ and let $f, g \in L^p_{\mathcal{K}}(\mathbb{R}^n)$ for $0 < p \leqslant 1$. Then*

$$\|f * g\|_{L^p(\mathbb{R}^n)} \lesssim r^{n\left(\frac{1}{p}-1\right)} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}\,.$$

In establishing the local boundedness of FIOs for the optimal ranges of p's, the following Bernstein-type estimate will be useful. The proof can be found in [23, p. 22].

Lemma 2.19. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact set and let $0 < p \leqslant r \leqslant \infty$. Then*

$$\|\partial^\alpha f\|_{L^r(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

for all multi-indices α and all $f \in L^p_{\mathcal{K}}(R^n)$.

In order to establish L^p estimates $(0 < p \leqslant 1)$ for a generic Littlewood-Paley piece of the FIOs, we will estimate the so called *Peetre's maximal functions* by the Hardy-Littlewood maximal operators as in the following lemma:

Lemma 2.20. *Let $\{f_{j,k}\} \in L^p_{\mathcal{K}_{j,k}}(\mathbb{R}^n)$ with*

$$\mathcal{K}_{j,k} := \Big\{(\xi_1, \xi_2) \in \mathbb{R}^{n'} \times \mathbb{R}^{n-n'} :\ |\xi_1| \leqslant c_{j,k}, |\xi_2| \leqslant d_{j,k}\Big\}$$

where $0 \leqslant n' \leqslant n$ and $c_{j,k}$ and $d_{j,k}$ are some positive constants. Futhermore, let $x := (x_1, x_2) \in \mathbb{R}^{n'} \times \mathbb{R}^{n-n'}$ and $z := (z_1, z_2) \in \mathbb{R}^{n'} \times \mathbb{R}^{n-n'}$. Then one has

$$\sup_{z\in\mathbb{R}^n} \frac{|f_{j,k}(z)|}{\Big(1 + (c_{j,k}|x_1 - z_1|)^{\frac{1}{r_1}}\Big)\Big(1 + (d_{j,k}|x_2 - z_2|)^{\frac{1}{r_2}}\Big)} \lesssim \Big(M_2\left(M_1 |f_{j,k}|^{r_1}\right)^{\frac{r_2}{r_1}}\Big)^{\frac{1}{r_2}}(x)$$

uniformly in j, k and for all $r_1, r_2 > 0$ small enough. Here M_1 is the Hardy-Littlewood maximal function acting on the function in the x_1-variable, i.e.

$$M_1 f(x) := \sup_{\delta>0} \frac{1}{|B(x_1,\delta)|} \int_{B(x_1,\delta)} |f(y_1, x_2)|\,\mathrm{d}y_1,$$

and M_2 is defined in a similar way.

Proof. Let $g \in L^p_{\mathcal{K}}(\mathbb{R}^n)$ for $\mathcal{K} := \big\{(\xi_1, \xi_2) \in \mathbb{R}^{n'} \times \mathbb{R}^{n-n'} :\ |\xi_1| \leqslant 1, |\xi_2| \leqslant 1\big\}$. Then

$$\sup_{z\in\mathbb{R}^n} \frac{|\partial^\alpha g(z)|}{\Big(1 + (|x_1 - z_1|)^{\frac{1}{r_1}}\Big)\Big(1 + (|x_2 - z_2|)^{\frac{1}{r_2}}\Big)} \lesssim \Big(M_2\left(M_1 |g|^{r_1}\right)^{\frac{r_2}{r_1}}\Big)^{\frac{1}{r_2}}(x). \tag{10}$$

The proof of (10) when $n = 2$ and $n' = 1$ can be found in [19, p. 48, Equation (1)]. By carefully tracing that proof, one can generalise the result to (10). Then the lemma follows by setting $f_{j,k}(x_1, x_2) := g(c_{j,k}x_1, d_{j,k}x_2)$ and $\alpha = 0$. □

Finally we state the following version of the non-stationary phase lemma, whose proof can be found in [17, Lemma 3.2].

Lemma 2.21. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact set and $\Omega \supset \mathcal{K}$ an open set. Assume that Φ is a real valued function in $\mathcal{C}^\infty(\Omega)$ such that $|\nabla\Phi| > 0$ and*

$$|\partial^\alpha \Phi| \lesssim |\nabla\Phi|, \quad \left|\partial^\alpha\left(|\nabla\Phi|^2\right)\right| \lesssim |\nabla\Phi|^2,$$

for all multi-indices α with $|\alpha| \geqslant 1$. Then, for any $F \in \mathcal{C}^\infty_c(\mathcal{K})$ and any integer $k \geqslant 0$,

$$\left|\int_{\mathbb{R}^n} F(\xi)\, e^{i\Phi(\xi)}\, \text{đ}\xi\right| \leqslant C_{k,n,\mathcal{K}} \sum_{|\alpha|\leqslant k} \int_{\mathcal{K}} |\partial^\alpha F(\xi)|\, |\nabla\Phi(\xi)|^{-k}\, \text{đ}\xi.$$

3. The Seeger-Sogge-Stein decomposition and the associated kernel estimates

In connection to the study of the L^p regularity of FIOs, A. Seeger. C. Sogge and E. Stein introduced a second dyadic decomposition superimposed on a preliminary Littlewood-Paley decomposition, in which each dyadic shell $\{2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}$ (as in Definition 2.1) is further partitioned into truncated cones of thickness roughly $2^{\frac{j}{2}}$ and one can prove that $O\left(2^{j\frac{n-1}{2}}\right)$ such elements are needed to cover one shell.

Definition 3.1. *For each $j \in \mathbb{N}$ we fix a collection of unit vectors $\left\{\xi_j^\nu\right\}$ that satisfy the following two conditions.*

(i) $\left|\xi_j^\nu - \xi_j^{\nu'}\right| \geqslant 2^{-\frac{j}{2}}$, *if $\nu \neq \nu'$.*

(ii) *If $\xi \in \mathbb{S}^{n-1}$, then there exists a ξ_j^ν so that $\left|\xi - \xi_j^\nu\right| < 2^{-\frac{j}{2}}$.*

One can take a collection $\{\xi_j^\nu\}$ which is maximal with respect to the first property and there are at most $O\left(2^{j\frac{n-1}{2}}\right)$ elements in the collection $\{\xi_j^\nu\}$.

Let Γ_j^ν denote the cone in the ξ space whose central direction is ξ_j^ν, i.e.

$$\Gamma_j^\nu := \left\{\xi \in \mathbb{R}^n : \left|\frac{\xi}{|\xi|} - \xi_j^\nu\right| \leqslant 2 \cdot 2^{-\frac{j}{2}}\right\}. \tag{11}$$

One also defines

$$\eta_j^\nu(\xi) := \phi\left(2^{\frac{j}{2}}\left(\frac{\xi}{|\xi|} - \xi_j^\nu\right)\right), \tag{12}$$

where ϕ is a nonnegative function in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\phi(u) = 1$ for $|u| \leqslant 1$ and $\phi(u) = 0$ for $u \geqslant 2$.

As was done in [20] one could set

$$\chi_j^\nu := \eta_j^\nu \left(\sum_\nu \eta_j^\nu\right)^{-1}$$

which is in $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ and supported in the cone Γ_j^ν satisfying the estimates

$$\left|\partial_\xi^\alpha \chi_j^\nu(\xi)\right| \lesssim 2^{j\frac{|\alpha|}{2}} |\xi|^{-j|\alpha|} \tag{13}$$

for all multi-indices α and

$$\left|\partial_{\xi_1}^N \chi_j^\nu(\xi)\right| \leqslant C_N |\xi|^{-N}, \quad \text{for} \quad N \geqslant 1, \tag{14}$$

if one chooses the axis in ξ-space such that ξ_1 is in the direction of ξ_j^ν and $\xi' = (\xi_2, \dots, \xi_n)$ is perpendicular to ξ_j^ν. With this construction, it is also clear that

$$\sum_\nu \chi_j^\nu(\xi) = 1, \quad \text{for all } j \text{ and } \xi \neq 0. \tag{15}$$

Therefore, if ψ_j is chosen as in Definition 2.1, one has

$$\psi_0(\xi) + \sum_{j=1}^\infty \sum_\nu \chi_j^\nu(\xi)\psi_j(\xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{16}$$

It is sometimes useful to use a slightly different partition of unity by setting

$$\tilde{\chi}_j^\nu := \eta_j^\nu \left(\sum_\nu \left(\eta_j^\nu\right)^2 \right)^{-\frac{1}{2}}, \tag{17}$$

which satisfies

$$\sum_\nu \tilde{\chi}_j^\nu(\xi)^2 = 1, \qquad \text{for all } j \text{ and } \xi \neq 0. \tag{18}$$

Once again, one can show that (13) and (14) are also satisfied for $\tilde{\chi}_j^\nu$.

Using the Littlewood-Paley localisation ψ_j and the second dyadic frequency localisation χ_j^ν, we have the following estimate for the localised high frequency part of the kernels:

Lemma 3.2. *Let $j \geqslant 1$ and set*

$$K_j^\nu(x,y) := \int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)} \psi_j(\xi) \chi_j^\nu(\xi) a(x,y,\xi) \, đ\xi \tag{19}$$

where $a \in L^\infty A^m(\mathbb{R}^n)$, $\Phi(x,y,\xi) = \varphi(x,\xi) - y\cdot\xi$ or $\Phi(x,y,\xi) = x\cdot\xi - \varphi(y,\xi)$ and $\varphi(x,\xi) \in \Phi^2$. Then for all $N \geqslant 0$, the kernel K_j^ν satisfies the estimate

$$\begin{aligned} |K_j^\nu(x,y)| \lesssim\, & 2^{j\left(m+\frac{n+1}{2}\right)} \left(1 + \left|2^j \nabla_{\xi_1}\Phi(x,y,\xi_j^\nu)\right|^2\right)^{-N} \\ & \times \left(1 + \left|2^{\frac{j}{2}} \nabla_{\xi'}\Phi(x,y,\xi_j^\nu)\right|^2\right)^{-N}. \end{aligned} \tag{20}$$

Proof. Define $h(x,y,\xi) := \Phi(x,y,\xi) - \nabla_\xi \Phi(x,y,\xi_j^\nu)\cdot\xi$. Then one has

$$K_j^\nu(x,y) = \int_{\mathbb{R}^n} e^{i\nabla_\xi\Phi(x,y,\xi_j^\nu)\cdot\xi}\, b_j^\nu(x,y,\xi)\, đ\xi,$$

where $b_j^\nu(x,y,\xi) := \psi_j(\xi)\chi_j^\nu(\xi) e^{ih(x,y,\xi)}$. It can be verified (see e.g. [21, p. 407]) that the phase $h(x,y,\xi)$ satisfies

$$\left|\partial_{\xi_1}^N h(x,y,\xi)\right| \leqslant C_N 2^{-jN} \tag{21}$$

$$\left|\partial_{\xi'}^{\alpha'} h(x,y,\xi)\right| \leqslant C_N 2^{-j\frac{|\alpha'|}{2}}, \tag{22}$$

for $N \geqslant 2$ on the support of $b_j^\nu(x,y,\xi)$. Introducing the differential operator $L := \left(I - 2^{2j}\partial_{\xi_1^2}^2\right)\left(I - 2^j \Delta_{\xi'}\right)$, one can check that

$$\begin{aligned} L^N\left(e^{i\nabla_\xi\Phi(x,y,\xi_j^\nu)\cdot\xi}\right) = e^{i\nabla_\xi\Phi(x,y,\xi_j^\nu)\cdot\xi} & \left(1 + \left|2^j \nabla_{\xi_1}\Phi(x,y,\xi_j^\nu)\right|^2\right)^N \\ \times & \left(1 + \left|2^{\frac{j}{2}} \nabla_{\xi'}\Phi(x,y,\xi_j^\nu)\right|^2\right)^N, \end{aligned}$$

Furthermore for $b_j^\nu(x,y,\xi)$ using the assumption that $a \in S^m(\mathbb{R}^n)$ together with (13), (14), and the uniform estimates (in x) for $h(x,y,\xi)$ in (21) and (22), we can show that for any $j \in \mathbb{N}$, ν and $\xi \in \operatorname{supp}_\xi b_j^\nu$

$$\left\|L^N b_j^\nu(\cdot,\cdot,\xi)\right\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant C_N 2^{jm}. \tag{23}$$

Now integration by parts yields

$$\begin{aligned}
&\left|K_j^\nu(x,y)\right| \leqslant \left(1+\left|2^j\nabla_{\xi_1}\Phi(x,y,\xi_j^\nu)\right|^2\right)^{-N}\\
&\times\left(1+\left|2^{\frac{j}{2}}\nabla_{\xi'}\Phi(x,y,\xi_j^\nu)\right|^2\right)^{-N}\int_{\operatorname{supp}_\xi\, b_j^\nu}\left|L^N b_j^\nu(x,y,\xi)\right|\,\text{đ}\xi\\
&\lesssim 2^{jm}2^{j\frac{n+1}{2}}\left(1+\left|2^j\nabla_{\xi_1}\Phi(x,y,\xi_j^\nu)\right|^2\right)^{-N}\left(1+\left|2^{\frac{j}{2}}\nabla_{\xi'}\Phi(x,y,\xi_j^\nu)\right|^2\right)^{-N}
\end{aligned}$$

where we used (23) and that $\left|\operatorname{supp} b_j^\nu\right| = O\left(2^{j\frac{n+1}{2}}\right)$. Hence the proof is complete. ☐

Remark 3.3. *The conclusion of* Lemma 3.2*, is also valid if the phase function φ is merely assumed to be in $\mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$ and positively homogeneous of degree one in ξ.*

Lemma 3.4. *For $j\geqslant 1$ and $0<p\leqslant 1$, let*

$$K_j^\nu(x,y) := \int_{\mathbb{R}^n}\sigma(x,y,\xi)\,\psi_j(\xi)\chi_j^\nu(\xi)\,e^{i\Phi(x,y,\xi)}\,\text{đ}\xi$$

where $\sigma\in L^\infty A^{m_c(p)}(\mathbb{R}^n)$, $\Phi(x,y,\xi)=\varphi(x,\xi)-y\cdot\xi$ or $\Phi(x,y,\xi)=x\cdot\xi-\varphi(y,\xi)$ and $\varphi\in\Phi^2$ satisfies the SND *condition* (8)*. Then one has*

(i) $\displaystyle\int_{\mathbb{R}^n}|K_j(x,y)|^p\,\mathrm{d}x\leqslant A2^{jn(p-1)}$ *uniformly in $y\in\mathbb{R}^n$,*

(ii) $\displaystyle\int_{\mathbb{R}^n}\sup_{y\in B(y',r)}|K_j(x,y)-p_j(x,y-y')|^p\,\mathrm{d}x\leqslant A\,2^{jn(p-1)}2^{jMp}\,r^{Mp}$*, where $p_j(x,y-y')$ is the Taylor polynomial of K_j centred at y' of order $M-1$, where $M=1+\left[n\left(\frac{1}{p}-1\right)\right]$.*

(iii) $\displaystyle\int_{B^{*c}}\sup_{y\in B(\overline{y},r)}|K_j(x,y)|^p\,\mathrm{d}x\leqslant A\,2^{jn(p-1)}2^{-j}r^{-1}$*, for $2^j\geqslant r^{-1}$, where B^* is a suitable "influence set" associated to the phase function $\Phi(x,y,\xi)$ (see equation* (24)*).*

In all the estimates above, the constant A is independent of y', $\overline{y}$, j and r.

Proof. To prove (i), we use Lemma 3.2 which yields that

$$\left|K_j^\nu(x,y)\right|\lesssim 2^{jm}2^{j\frac{n+1}{2}}\left(1+\left|2^j\nabla_{\xi_1}\Phi(x,y,\xi_j^\nu)\right|^2\right)^{-N}\left(1+\left|2^{\frac{j}{2}}\nabla_{\xi'}\Phi(x,y,\xi_j^\nu)\right|^2\right)^{-N}.$$

We raise K_j^ν to the power p and integrate the above inequality in $x\in\mathbb{R}^n$. In doing that we also use the change of variables $z=\nabla_\xi\varphi(x,\xi_j^\nu)$, in the case of $\Phi(x,y,\xi)=\varphi(x,\xi)-y\cdot\xi$ whose modulus of the Jacobian determinant is uniformly bounded from below by a non-zero constant (by the SND condition). Note that no changes of variables is needed when $\Phi(x,y,\xi) = -\varphi(y,\xi) + x\cdot\xi$. At any rate, this yields for all $y \in \mathbb{R}^n$ that

$$\int_{\mathbb{R}^n} |K_j^\nu(x,y)|^p \,\mathrm{d}x \leqslant C_N \int_{\mathbb{R}^n} \frac{2^{j\frac{p(2m+n+1)}{2}}}{\left(1+\left|2^j z_1\right|^2+\left|2^{\frac{j}{2}} z'\right|^2\right)^N}\,\mathrm{d}z$$
$$\lesssim \int_{\mathbb{R}^n} \frac{2^{j\frac{p(2m+n+1)}{2}}\,2^{-j\frac{2+(n-1)}{2}}}{(1+|z|^2)^N}\,\mathrm{d}z \lesssim 2^{jn(p-1)}\,2^{-j\frac{n-1}{2}}.$$

Therefore summing in ν and remembering that there are $O\left(2^{j\frac{n-1}{2}}\right)$ terms involved, we obtain

$$\int_{\mathbb{R}^n} |K_j(x,y)|^p \,\mathrm{d}x \lesssim \sum_\nu 2^{jn(p-1)} 2^{-j\frac{n-1}{2}} \lesssim 2^{jn(p-1)},$$

for all $y \in \mathbb{R}^n$; this proves (i) in Lemma 3.4.

The proof of (ii) is rather similar to that of (i). Here one can show that for all multi-indices α and all $y \in \mathbb{R}^n$ one has

$$\int_{\mathbb{R}^n} |\partial_y^\alpha K_j^\nu(x,y)|^p \,\mathrm{d}x \lesssim 2^{|\alpha|jp}\,2^{jn(p-1)}\,2^{-j\frac{n-1}{2}}.$$

If we let $p_j^\nu(x, y-y')$ be the Taylor polynomial of $K_j^\nu(x,y)$ of order $M-1$ centred at $y=y'$, then we have

$$\sup_{y\in B(y',r)} \int_{\mathbb{R}^n} |K_j^\nu(x-y) - p_j^\nu(x,y-y')|^p \,\mathrm{d}x \lesssim 2^{jMp}\,2^{jn(p-1)}\,2^{-j\frac{n-1}{2}} r^{Mp}$$

and summing over ν's we obtain (ii).

Now let B be the ball of centre $\overline{y}$ and radius $r \leqslant 1$. Define the "rectangles" R_j^ν by

$$R_j^\nu = \left\{x \in \mathbb{R}^n \;:\; |\nabla_\xi \Phi(x,\overline{y},\xi_j^\nu)| \leqslant A2^{-\frac{j}{2}},\ |\pi_j^\nu(\nabla_\xi\Phi(x,\overline{y},\xi_j^\nu))| \leqslant A2^{-j}\right\}.$$

where π_j^ν is the orthogonal projection in the direction ξ_j^ν. Note that our Φ here is of the specific forms given in the statement of Lemma 3.2. If one defines

$$B^* = \bigcup_{2^{-j}\leqslant r} \bigcup_\nu R_j^\nu \tag{24}$$

then one can show (see e.g. [21]) that

$$|B^*| \lesssim r.$$

Now to prove (iii), let $r \geqslant 2^{-j}$, and suppose that k is an integer such that $2^{-k} \leqslant r \leqslant 2^{-k+1}$. Then there is a unit vector ξ_k^μ such that $\left|\xi_j^\nu - \xi_k^\mu\right| \leqslant 2^{-\frac{k}{2}}$. Using the definition of R_k^μ we have for $x \notin R_k^\mu$

$$\left|2^k \pi_k^\mu(\nabla_\xi \Phi(x,\overline{y},\xi_k^\mu))\right| + \left|2^{\frac{k}{2}}\nabla_\xi\Phi(x,\overline{y},\xi_k^\mu)\right| \geqslant C,$$

for a sufficiently large C. Therefore, there exists a constant $C>0$ such that for every $y \in B$ and $x \in B^{*c}$ we have

$$\left|2^j\nabla_{\xi_1}\Phi(x,y,\xi_j^\nu))\right| + \left|2^{\frac{j}{2}}\nabla_{\xi'}\Phi(x,y,\xi_j^\nu)\right| \geqslant C2^{j-k}.$$

Therefore changing variables as before, we see that for $y \in B$ and $2^{-j} \leqslant r$ one has

$$
\begin{aligned}
&\int_{B^{*c}} \sup_{y\in B} \left|K_j^\nu(x,y)\right|^p \,\mathrm{d}x \\
&\lesssim \int_{B^{*c}} \frac{2^{j\frac{p(2m+n+1)}{2}}}{\left(1+\left|2^j\nabla_{\xi_1}\Phi(x,y,\xi_j^\nu)\right|^2+\left|2^{\frac{j}{2}}\nabla_{\xi'}\Phi(x,y,\xi_j^\nu)\right|^2\right)^N} \,\mathrm{d}x \\
&\lesssim \int_{\mathbb{R}^n} \frac{2^{j\frac{p(2m+n+1)}{2}}\, 2^{k-j}}{\left(1+\left|2^j z_1\right|^2+\left|2^{\frac{j}{2}}z'\right|^2\right)^{N-1}} \,\mathrm{d}z \\
&\lesssim \int_{\mathbb{R}^n} \frac{2^{j\frac{p(2m+n+1)}{2}}\, 2^{-j\frac{2+(n-1)}{2}}\, 2^{k-j}}{(1+|x|^2)^{N-1}} \,\mathrm{d}x \lesssim 2^{jn(p-1)}\, 2^{-j\frac{n-1}{2}}\, 2^{-j}\, r^{-1}.
\end{aligned}
$$

Therefore, summing once again in ν yields (iii). □

4. Ruzhansky-Sugimoto's globalisation technique

In [18], M. Ruzhansky and M. Sugimoto developed a new technique to transfer local boundedness of Fourier integral operators, which was proven by C. Sogge, A. Seeger and E. Stein [20], to a global result, where the amplitudes of the corresponding operators do not have compact spatial supports. In order to prove global regularity results we follow [18] and define

$$H(x,z) = \inf_{\xi\in\mathbb{R}^n} |z+\nabla_\xi \vartheta(x,y,\xi)|$$

where for us $\vartheta(x,y,\xi)$ is either $\theta(x,\xi)$ or $-\theta(y,\xi)$, with $\theta \in \Phi^1$ and

$$\Delta_r := \{(x,y,z)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n : H(x,y,z)\geqslant r\}.$$

One also defines

$$
\begin{aligned}
\widetilde{H}(z) &:= \inf_{x\in\mathbb{R}^n} H(x,y,z) = \inf_{x,y,\xi\in\mathbb{R}^n} |z+\nabla_\xi\vartheta(x,y,\xi)|\,, \\
\widetilde{\Delta}_r &:= \{z\in\mathbb{R}^n : \widetilde{H}(z)\geqslant r\}.
\end{aligned}
$$

and

$$
\begin{aligned}
M_K &:= \sum_{|\gamma|\leqslant K} \sup_{x,y,\xi\in\mathbb{R}^n} \left|\langle\xi\rangle^{-(m_c-|\gamma|)}\,\partial_\xi^\gamma \sigma(x,y,\xi)\right|, \\
N_K &:= \sum_{1\leqslant|\gamma|\leqslant K} \sup_{x,y,\xi\in\mathbb{R}^n} \left|\langle\xi\rangle^{-(1-|\gamma|)}\,\partial_\xi^\gamma \vartheta(x,y,\xi)\right|.
\end{aligned}
$$

We observe that $N_K<\infty$ by the Φ^1 condition on the phase function. Given these definitions one has the following lemma:

Lemma 4.1. *Let $r\geqslant 1$ and $K\geqslant 1$. Then we have $\mathbb{R}^n\setminus\widetilde{\Delta}_{2r}\subset\{z;\ |z|<(2+N_K)r\}$. Furthermore for $r>0$, $x\in\widetilde{\Delta}_{2r}$ and $|y|\leqslant r$ we have*

$$\widetilde{H}(x)\leqslant 2H(x,y,x-y) \tag{25}$$

and therefore $(x,y,x-y)\in\Delta_r$

Proof. For $z \in \mathbb{R}^n \setminus \widetilde{\Delta}_{2r}$, we have $\widetilde{H}(z) < 2r$. Hence, there exist $x_0, y_0, \xi_0 \in \mathbb{R}^n$ such that

$$|z + \nabla_\xi \vartheta(x_0, y_0, \xi_0)| < 2r.$$

Since, $r \geqslant 1$, this yields that

$$|z| \leqslant |z + \nabla_\xi \vartheta(x_0, y_0, \xi_0)| + |\nabla_\xi \vartheta(x_0, y_0, \xi_0)| \leqslant 2r + N_K \leqslant (2 + N_K)r.$$

The claim that $(x, y, x-y) \in \Delta_r$ follows from (25) and the definition of Δ_r. Therefore it only remains to prove (25). Now, if $|y| \leqslant r$ and $x \in \widetilde{\Delta}_{2r}$ then since $\widetilde{H}(x) \geqslant 2r$, we have that

$$\begin{aligned}\widetilde{H}(x) &\leqslant |x + \nabla\vartheta(x, y, \xi)| \leqslant |x - y + \nabla\vartheta(x, y, \xi)| + |y| \\ &\leqslant |x - y + \nabla\vartheta(x, y, \xi)| + \frac{\widetilde{H}(x)}{2}.\end{aligned}$$

From this, (25) follows at once. □

For proving the global boundedness that we aim to demonstrate, the following result, is of particular importance.

Lemma 4.2. *The kernel*

$$K(z) = \int_{\mathbb{R}^n} e^{iz\cdot\xi + i\vartheta(x,y,\xi)}\, \sigma(x, y, \xi)\, đ\xi$$

is smooth on $\bigcup_{r>0} \Delta_r$*, and for all* $L > n$ *it satisfies*

$$\left\| H^L K \right\|_{L^\infty(\Delta_r)} \leqslant C(r, L, M_L, N_{L+1}), \tag{26}$$

where $C(r, L, M_L, N_{L+1})$ *is a positive constant depending only on* L*,* $r > 0$*,* M_L *and* N_{L+1}*. For* $0 < p \leqslant 1$ *and* $L > \dfrac{n}{p}$*, the function* $\widetilde{H}(z)$ *satisfies the bound*

$$\left\| \widetilde{H}^{-L} \right\|_{L^p\left(\widetilde{\Delta}_r\right)} \leqslant C(r, L, N_{L+1}, p). \tag{27}$$

Proof. If one introduces the differential operator

$$D = \frac{(z + \nabla\vartheta) \cdot \nabla_\xi}{i|z + \nabla\vartheta|^2},$$

with the transpose D^*, then integration by parts L times yields

$$K(z) = \int_{\mathbb{R}^n} e^{iz\cdot\xi + i\vartheta(x,y,\xi)}\, (D^*)^L\, \sigma(x, y, \xi)\, đ\xi.$$

Now (26) follows from the relation $r \leqslant H(x, y, z) \leqslant |z + \nabla_\xi \vartheta(x, y, \xi)|$ which is valid for $(x, y, z) \in \Delta_r$ and $\xi \in \mathbb{R}^n$. Moreover $|z| \leqslant |z + \nabla_\xi \vartheta(x, y, \xi)| + N_{L+1}$ for any $\xi \neq 0$, which yields that $|z| \leqslant \widetilde{H}(z) + N_{L+1}$. Hence for $|z| \geqslant 2N_{L+1}$ one has $|z| \leqslant \widetilde{H}(z) + \dfrac{|z|}{2}$, and therefore $|z| \leqslant 2\widetilde{H}(z)$. Using this we get

$$\begin{aligned}\left\|\widetilde{H}^{-L}\right\|_{L^p\left(\widetilde{\Delta}_r\right)} &\leqslant \left\|\widetilde{H}^{-L}\right\|_{L^p\left(\widetilde{\Delta}_r\cap\{|z|\leqslant 2N_{L+1}\}\right)} + \left\|\widetilde{H}^{-L}\right\|_{L^p\left(\widetilde{\Delta}_r\cap\{|z|\geqslant 2N_{L+1}\}\right)} \\ &\leqslant r^{-L}\left(\int_{|z|\leqslant 2N_{L+1}} \mathrm{d}z\right)^{\frac{1}{p}} + 2^L\left(\int_{|z|\geqslant 2N_{L+1}} |z|^{-pL}\,\mathrm{d}z\right)^{\frac{1}{p}} \\ &\leqslant C(r, L, N_{L+1}),\end{aligned}$$

which proves (27) . □

Now in the proof of global boundedness of FIOs that are treated in this paper, we shall use Lemma 2.15 to bring the operators in question to the form

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,\xi)\, e^{i\theta(x,\xi)+i(t(x)-y)\cdot\xi}\, u(y)\,\text{đ}\xi\,\mathrm{d}y,$$

or

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} a(y,\xi)\, e^{i\theta(y,\xi)+i(t(y)-x)\cdot\xi}\, u(y)\,\text{đ}\xi\,\mathrm{d}y,$$

where $\theta \in \Phi^1$, and $t(\cdot)$ is an appropriate global diffeomorphism. Therefore a change of variables and using the invariance of Besov-Lipschitz and Triebel-Lizorkin spaces under suitable diffeomorphisms, will enable us to replace $t(x)$ and $t(y)$ by x and y respectively and utilise the estimates discussed above, to obtain global boundedness results in various settings.

5. Boundedness of FIOs on Besov-Lipschitz spaces

In this section we establish the boundedness of FIO's of all possible scales for Besov-Lipschitz spaces $B^s_{p,q}(\mathbb{R}^n)$ for $-\infty < s < \infty$, $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$. The local boundedness results are for amplitudes $a(x,\xi) \in S^m(\mathbb{R}^n)$ and phase functions $\varphi(x,\xi)$ that are positively homogeneous of degree 1 in ξ and satisfy the usual non-degeneracy condition. We will also prove global boundedness results for operators with phase functions in Φ^2 that are SND. For the global results to hold, it is necessary that $p > \dfrac{n}{n+1}$. At this point, it is appropriate to note that the phase function of the Fourier integral operators are in general singular at the origin, therefore in proving various boundedness results, it behoves one to split the operator in high and low frequency parts. Henceforth we shall divide the regularity results into low and high frequency portions.

5.1. **L^p boundedness of a Littlewood-Paley piece of a Fourier integral operator.** Briefly, the result concerning the L^p boundedness of the Littlewood-Paley pieces of an FIO states that, if the operator in question has an amplitude with frequency support in an annulus of size $\sim 2^j$, $j \in \mathbb{N}$, then that operator is L^p bounded. Moreover, the L^p estimate keeps control of the parameter j. This will be crucial when estimating the L^p norm o an FIO within the $B^s_{p,q}$ norm.

Proposition 5.1. *Let* $0 < p \leqslant \infty$, $m_c(p)$ *as in* (5), $a \in L^\infty S^m(\mathbb{R}^n)$ *and* $\varphi(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$, *be positively homogeneous of degree one in* ξ. *Assume that* ψ_j *is as in* Definition 2.1 *and let* T_j *be a Littlewood-Paley piece of an* FIO T_a^φ, *which is defined by*

$$T_j f(x) := \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi)(1-\psi_0(2\xi))\psi_j(\xi)\widehat{f}(\xi)\,đ\xi. \tag{28}$$

Then if $\varphi \in \Phi^2$ *is* SND, *one has*

$$\|T_j f\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j(m-m_c(p))} \|\Psi_j(D)f\|_{L^p(\mathbb{R}^n)}, \tag{29}$$

for $j \in \mathbb{N}$ *and* Ψ_j *as defined in* Definition 2.1. *Furthermore, if one assumes that the amplitude* $a(x,\xi)$ *is compactly supported in* x, *then one has the same result, if the phase function* φ *is assumed to be non-degenerate on the support of* $a(x,\xi)$.

Remark 5.2. *The factor* $(1-\psi_0(2\xi))$ *is inserted in* (28) *to cut off the singularity at* $\xi = 0$ *for the case* $j = 0$. *The singularity has to be taken care of separately and this is done in propositions* 5.5, 5.6 *below.*

Remark 5.3. *Note that in the Banach cases, i.e.* $p \in [1,\infty]$, (29) *is equivalent to the* L^p *boundedness of operators* T_j. *However in the quasi-Banach cases, i.e* $p < 1$, *then one can not get rid of the frequency localisation* $\Psi_j(D)$, *since any* L^p *bounded translation invariant operator* (*for* $0 < p < 1$) *is an infinite linear combination* (*with coefficients in* ℓ^p) *of Dirac measures, see* [12].

Proof of Proposition 5.1. Since the proof is rather lengthy and contains several cases, we split it into four steps as follows;

- (i) In Step 1 we use the kernel estimate from Lemma 3.2 and prove the proposition for the case $0 < p \leqslant 1$.
- (ii) In Step 2 we once again use Lemma 3.2 to obtain the result for $p = \infty$.
- (iii) In Step 3 we deal with the case of $p = 2$.
- (iv) In Step 4 we show the result for the cases $1 < p < 2$ and $2 < p < \infty$, and finally interpolation yields the boundedness for the range $1 < p \leqslant \infty$.

Note that in the proofs of (ii), (iii) and (iv), it will be enough to show an estimate of the form

$$\|T_j f\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j(m-m_c(p))} \|f\|_{L^p(\mathbb{R}^n)},$$

where we could without any cost, insert a frequency localisation on the right hand side of the estimate above.

Step 1 – Proof of the case $0 < p \leqslant 1$

We will use the partition of unity (18) and decompose the operator T_j as $T_j = \sum_\nu T_j^\nu$, where

$$\begin{aligned}
T_j^\nu f(x) &:= \iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,\xi)\,(1-\psi_0(2\xi))\,\psi_j(\xi)\,\tilde{\chi}_j^\nu(\xi)^2\, e^{i\varphi(x,\xi)-iy\cdot\xi}\, f(y)\,đ\xi\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} K_j^\nu(x,y)\,\mathcal{X}_j^\nu(D)\,\Psi_j(D)\,f(y)\,\mathrm{d}y, \\
K_j^\nu(x,y) &:= \int_{\mathbb{R}^n} a(x,\xi)\,(1-\psi_0(2\xi))\,\psi_j(\xi)\,\tilde{\chi}_j^\nu(\xi)\, e^{i\varphi(x,\xi)-iy\cdot\xi}\,đ\xi,
\end{aligned}$$

where $\mathcal{X}_j^\nu(D) := \tilde{\chi}_j^\nu(D)\,\Psi_j(D)$ with $\tilde{\chi}_j^\nu$ as in (17) and $\Psi_j(D)$ as in Definition 2.1. Using the properties (13) and (14) which are also valid for $\tilde{\chi}_j^\nu$, one can verify that the kernel K_j^ν satisfies (20). Now set $f_j^\nu := \mathcal{X}_j^\nu(D)\Psi_j(D)f$.
At this point, we set

$$\begin{aligned}\mathsf{f}_j^\nu(z) &:= \sup_{y\in\mathbb{R}^n}\Big(1+\big|2^j(z_1-y_1)\big|^2\Big)^{-M}\Big(1+\big|2^{\frac{j}{2}}(z'-y')\big|^2\Big)^{-M}\big|f_j^\nu(y)\big|\\ &\lesssim \sup_{y\in\mathbb{R}^n}\Big(1+\big|2^j(z_1-y_1)\big|^{2M}\Big)^{-1}\Big(1+\big|2^{\frac{j}{2}}(z'-y')\big|^{2M}\Big)^{-1}\big|f_j^\nu(y)\big|.\end{aligned}$$

Since $\Psi_j\equiv 1$ on the support of ψ_j we have (using (20))

$$\begin{aligned}\big|T_j^\nu f(x)\big| &\leqslant \int_{\mathbb{R}^n}\big|K_j^\nu(x,y)f_j^\nu(y)\big|\,\mathrm{d}y\\ &\lesssim 2^{jm}2^{j\frac{n+1}{2}}\mathsf{f}_j^\nu(\nabla_\xi\varphi(x,\xi_j^\nu))\int_{\mathbb{R}^n}\Big(1+\big|2^j(\nabla_{\xi_1}\varphi(x,\xi_j^\nu)-y_1)\big|^2\Big)^{M-N}\\ &\times\Big(1+\big|2^{\frac{j}{2}}(\nabla_{\xi'}\varphi(x,\xi_j^\nu)-y')\big|^2\Big)^{M-N}\,\mathrm{d}y\\ &\lesssim 2^{jm}2^{j\frac{n+1}{2}}2^{-j\frac{n+1}{2}}\mathsf{f}_j^\nu(\nabla_\xi\varphi(x,\xi_j^\nu)) = 2^{jm}\mathsf{f}_j^\nu(\nabla_\xi\varphi(x,\xi_j^\nu)),\end{aligned}$$

where $M>\dfrac{1}{2p}$, $N-M>n$.

Now in Lemma 2.20, take $k=\nu$, $n'=1$ and $r_1=r_2=\dfrac{1}{2M}<p$ and note that $\operatorname{supp}\widehat{f_j^\nu}\subset\Big\{(\xi_1,\xi')\in\mathbb{R}\times\mathbb{R}^{n-1}:\ |\xi_1|\leqslant 2^j,|\xi'|\leqslant 2^{\frac{j}{2}}\Big\}$. Moreover take $c_{j,\nu}=2^j$ and $d_{j,\nu}=2^{\frac{j}{2}}$. Then the conditions of Lemma 2.20 all hold for f_j^ν and therefore we have

$$\big|T_j^\nu f(x)\big|\leqslant 2^{jm}\mathsf{f}_j^\nu(\nabla_\xi\varphi(x,\xi_j^\nu))\lesssim 2^{jm}\Big(M_2\big(M_1|f_j^\nu|^{r_1}\big)^{\frac{r_2}{r_1}}\Big)^{\frac{1}{r_2}}\big(\nabla_\xi\varphi(x,\xi_j^\nu)\big).$$

Taking the L^p norm of the expression above, and using the SND condition on the phase function and changes of variables, the boundedness of the maximal operators M_1 and M_2 yields that

$$\begin{aligned}\big\|T_j^\nu f\big\|_{L^p(\mathbb{R}^n)} &\lesssim 2^{jm}\Big\|M_2(M_1|f_j^\nu|^{r_1})^{\frac{r_2}{r_1}}\Big\|^{\frac{1}{r_2}}_{L^{\frac{p}{r_2}}(\mathbb{R}^n)}\\ &\lesssim 2^{jm}\big\|(M_1|f_j^\nu|^{r_1})\big\|^{\frac{1}{r_1}}_{L^{\frac{p}{r_1}}(\mathbb{R}^n)}\lesssim 2^{jm}\big\|f_j^\nu\big\|_{L^p(\mathbb{R}^n)}.\end{aligned}\tag{30}$$

Here we observe that f_j^ν can be written of the form $f_j^\nu(x)=\big((\mathcal{X}_j^\nu)^\vee*\Psi_j(D)f\big)(x)$. Therefore, Lemma 2.18 yields

$$\big\|f_j^\nu\big\|_{L^p(\mathbb{R}^n)}\lesssim 2^{jn\left(\frac{1}{p}-1\right)}\big\|(\mathcal{X}_j^\nu)^\vee\big\|_{L^p(\mathbb{R}^n)}\|\Psi_j(D)f\|_{L^p(\mathbb{R}^n)}.\tag{31}$$

Now we would like to estimate $\big\|(\mathcal{X}_j^\nu)^\vee\big\|_{L^p(\mathbb{R}^n)}$. Indeed, using (13) and (14), integration by parts N times yields

$$\Big(1+\big|2^jz_1\big|^2\Big)^N\Big(1+\big|2^{\frac{j}{2}}z'\big|^2\Big)^N\Big|\big(\mathcal{X}_j^\nu\big)^\vee(z)\Big|\lesssim\int_{\operatorname{supp}\mathcal{X}_j^\nu}\big|L^N\mathcal{X}_j^\nu(\xi)\big|\,\text{đ}\xi\lesssim 2^{j\frac{n+1}{2}},$$

where we have used that $\left|\operatorname{supp}\mathcal{X}_j^\nu\right| = O\left(2^{j\frac{n+1}{2}}\right)$. Hence, it follows that

$$
\begin{aligned}
\left\|(\mathcal{X}_j^\nu)^\vee\right\|_{L^p(\mathbb{R}^n)} &\lesssim 2^{j\frac{n+1}{2}} \left(\int_{\mathbb{R}^n} \frac{1}{\left(\left(1+\left|2^j z_1\right|^2\right)^N \left(1+\left|2^{\frac{j}{2}} z'\right|^2\right)^N\right)^p}\,\mathrm{d}z\right)^{\frac{1}{p}} \\
&= 2^{j\frac{n+1}{2}} \left(\int_{\mathbb{R}^n} \frac{2^{-j}2^{-j\frac{n-1}{2}}}{\left(\left(1+\left|z_1\right|^2\right)^N \left(1+\left|z'\right|^2\right)^N\right)^p}\,\mathrm{d}z\right)^{\frac{1}{p}} \qquad (32)\\
&\lesssim 2^{j\left(\frac{n+1}{2}-\frac{n+1}{2p}\right)},
\end{aligned}
$$

for $N > n$. Inserting (32) in (31) and then (31) into (30) one has

$$
\begin{aligned}
\left\|T_j^\nu f\right\|_{L^p(\mathbb{R}^n)} &\lesssim 2^{jm} 2^{jn\left(\frac{1}{p}-1\right)} 2^{j\left(\frac{n+1}{2}-\frac{n+1}{2p}\right)} \left\|\Psi_j(D) f\right\|_{L^p(\mathbb{R}^n)} \\
&= 2^{j\left(m+\frac{n-1}{2p}-\frac{n-1}{2}\right)} \left\|\Psi_j(D) f\right\|_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

Summing in ν (note that there are $O\left(2^{j\frac{n-1}{2}}\right)$ terms involved)

$$
\begin{aligned}
\left\|T_j f\right\|_{L^p(\mathbb{R}^n)} &\leqslant \left(\sum_\nu \left\|T_j^\nu f\right\|^p_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{p}} \\
&\lesssim \left(\sum_\nu 2^{j\left(mp+\frac{n-1}{2}-p\frac{n-1}{2}\right)} \left\|\Psi_j(D) f\right\|^p_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{p}} \\
&\lesssim 2^{j\left(m+\frac{n-1}{p}-\frac{n-1}{2}\right)} \left\|\Psi_j(D) f\right\|_{L^p(\mathbb{R}^n)} \\
&= 2^{j(m-m_c(p))} \left\|\Psi_j(D) f\right\|_{L^p(\mathbb{R}^n)},
\end{aligned}
$$

and hence the proposition is proven for $0 < p \leqslant 1$.

Step 2 – Proof of the case $\mathbf{p = \infty}$
Once again we decompose $\mathbb{R}^n$ into cones as in Definition 3.1. This time the partition of unity χ_j^ν defined in (15). We then decompose T_j as $T_j = \sum_\nu T_j^\nu$, where

$$
T_j^\nu f(x) := \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)}\,(1-\psi_0(2\xi))\,\psi_j(\xi)\,\chi_j^\nu(\xi)\,a(x,\xi)\,\widehat{f}(\xi)\,\text{đ}\xi = \int_{\mathbb{R}^n} K_j^\nu(x,y) f(y)\,\mathrm{d}y,
$$

for

$$
K_j^\nu(x,y) := \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)-iy\cdot\xi}\,(1-\psi_0(2\xi))\,\psi_j(\xi)\,\chi_j^\nu(\xi)\,a(x,\xi)\,\text{đ}\xi.
$$

This yields

$$
\left|T_j^\nu(x)\right| \leqslant \left\|K_j^\nu(x,\cdot)\right\|_{L^1(\mathbb{R}^n)} \left\|f\right\|_{L^\infty(\mathbb{R}^n)} \qquad (33)
$$

Once again we have that K_j^ν satisfies (20), and by a change of variables

$$\left\|K_j^\nu(x,\cdot)\right\|_{L^1(\mathbb{R}^n)} \lesssim 2^{j\left(m+\frac{n+1}{2}\right)} \int_{\mathbb{R}^n} \left(1+\left|2^j(\nabla_{\xi_1}\varphi(x,\xi_j^\nu)-y_1)\right|^2\right)^{-N} \times \left(1+\left|2^{\frac{j}{2}}(\nabla_{\xi'}\varphi(x,\xi_j^\nu)-y')\right|^2\right)^{-N} \mathrm{d}y \lesssim 2^{jm}.$$

Hence the left hand side of (33) is independent of x and bounded by $2^{jm}\|f\|_{L^\infty(\mathbb{R}^n)}$. Using the fact that there are roughly $O\left(2^{j\frac{n-1}{2}}\right)$ terms in the sum in ν,

$$\|T_jf\|_{L^\infty(\mathbb{R}^n)} \lesssim \sum_\nu \left\|T_j^\nu f\right\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{j\left(m+\frac{n-1}{2}\right)}\|f\|_{L^\infty(\mathbb{R}^n)} = 2^{j(m-m_c(\infty))}\|f\|_{L^\infty(\mathbb{R}^n)}$$

and hence the proposition, when $p=\infty$, is proven.

Step 3 – Proof of the case p = 2
We proceed by studying the boundedness of $S_j := T_jT_j^*$. A simple calculation shows that $S_jf(x)=\displaystyle\int_{\mathbb{R}^n} K_j(x,y)f(y)\,\mathrm{d}y$ with

$$K_j(x,y) := \int_{\mathbb{R}^n} e^{i(\varphi(x,\xi)-\varphi(y,\xi))}\,(1-\psi_0(2\xi))^2\,\psi_j(\xi)^2\,a(x,\xi)\,\overline{a(y,\xi)}\,\text{đ}\xi.$$

Now since φ is homogeneous of degree 1 in the ξ variable, $K_j(x,y)$ can be written as

$$K_j(x,y) = 2^{jn}\int_{\mathbb{R}^n} b_j(x,y,2^j\xi)e^{i2^j\Phi(x,y,\xi)}\,\text{đ}\xi.$$

with

$$\begin{aligned}\Phi(x,y,\xi) &:= \varphi(x,\xi)-\varphi(y,\xi),\\ b_j\,(x,y,\xi) &:= (1-\psi_0(2\xi))^2\,\psi_j(\xi)^2\,a(x,\xi)\,\overline{a(y,\xi)}.\end{aligned}$$

Observe that the ξ-support of $b_j(x,y,2^j\xi)$ lies in the compact set $\mathcal{K} := \left\{\frac{1}{2}\leqslant|\xi|\leqslant 2\right\}$. From the SND condition (8) it also follows that

$$|\nabla_\xi\Phi(x,y,\xi)| \approx |x-y|, \quad \text{for any } x,y\in\mathbb{R}^n \text{ and } \xi\in\mathcal{K}. \tag{34}$$

Assume that $M>n$ is an integer, fix $x\neq y$ and set $\phi(\xi):=\Phi(x,y,\xi)$, $\Psi:=|\nabla_\xi\phi|^2$. By the mean value theorem, (6) and (34), for any multi-index α with $|\alpha|\geqslant 1$ and any $\xi\in\mathcal{K}$,

$$\left|\partial_\xi^\alpha\phi(\xi)\right| \lesssim |\nabla_\xi\Phi(x,y,\xi)| = \Psi^{\frac{1}{2}}.$$

On the other hand, since $\partial_\xi^\alpha\Psi = \displaystyle\sum_{j=1}^n\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\partial_\xi^\beta\partial_{\xi_j}\phi\,\partial_\xi^{\alpha-\beta}\partial_{\xi_j}\phi$, it follows that, for any $|\alpha|\geqslant 0$, $\left|\partial_\xi^\alpha\Psi\right|\lesssim\Psi$. We estimate the kernel K_j using in two different ways. For the first estimate, (34) and Lemma 2.21 with $F = b_j(x, y, 2^j\xi)$, yield

$$
\begin{aligned}
&|K_j(x,y)| \\
&\leqslant 2^{jn}2^{-jM}\, C_{M,\mathcal{K}} \sum_{|\alpha|\leqslant M} 2^{j|\alpha|} \int_{\mathbb{R}^n} \left|\partial_\xi^\alpha b_j(x,y,2^j\xi)\right| \left|\nabla_\xi \Phi(x,y,\xi)\right|^{-M} \,đ\xi \\
&\lesssim 2^{-jM} |x-y|^{-M} \sum_{|\alpha|\leqslant M} 2^{j|\alpha|} \int_{\mathbb{R}^n} \left|\partial_\xi^\alpha b_j(x,y,\xi)\right| \,đ\xi \\
&\lesssim 2^{j(2m+n)} \left(2^j |x-y|\right)^{-M}.
\end{aligned} \tag{35}
$$

where the fact that the ξ support of b_j lies in a ball of radius $\sim 2^j$ and that

$$
\left|\partial_\xi^\alpha b_j(x,y,\xi)\right| \lesssim 2^{j(2m-|\alpha|)}, \tag{36}
$$

have been used. Using (36) we also obtain

$$
|K_j(x,y)| \leqslant \int_{\mathbb{R}^n} |m_j(x,y,\xi)| \,đ\xi \lesssim 2^{j(2m+n)}, \tag{37}
$$

and when combining estimates (35) and (37) one has

$$
|K_j(x,y)| \lesssim 2^{j(2m+n)} \left(1 + 2^j |x-y|\right)^{-M}. \tag{38}
$$

Thus, using (38) and Minkowski's inequality we have

$$
\|S_j f\|_{L^2(\mathbb{R}^n)} \lesssim 2^{j(2m+n)} \left\| \int_{\mathbb{R}^n} \left(1+2^j|y|\right)^{-M} f(\,\cdot - y)\,\mathrm{d}y \right\|_{L^2(\mathbb{R}^n)} \lesssim 2^{2jm} \|f\|_{L^2(\mathbb{R}^n)}.
$$

Since $m_c(2) = 0$, the Cauchy-Schwarz inequality yields

$$
\left\|T_j^* f\right\|_{L^2(\mathbb{R}^n)}^2 = \left\langle T_j T_j^* f, f\right\rangle_{L^2(\mathbb{R}^n)} \lesssim \|S_j f\|_{L^2(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)} = 2^{2j(m-m_c(2))} \|f\|_{L^2(\mathbb{R}^n)}^2.
$$

Therefore $\|T_j\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = \|T_j^*\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim 2^{j(m-m_c(2))}$ and the proposition is proven for the case $p = 2$.

Step 4 – Proof of the case $\mathbf{1 < p < 2}$ and $\mathbf{2 < p < \infty}$
Now that we have the desired result for $p = 1$, $p = 2$ and $p = \infty$, we can complete the proof of the proposition. Indeed, the Riesz-Thorin interpolation theorem in $1 < p < 2$ and $2 \leqslant p \leqslant \infty$ yields that

$$
\|T_j f\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j(m-m_c(p))} \|f\|_{L^p(\mathbb{R}^n)},
$$

which thereby concludes the proof of Proposition 5.1, when the amplitude is not compactly supported in x.

In case $a(x,\xi)$ is compactly supported in x, then the homogeneity of the phase, and its non-degeneracy will once again yield all the kernel estimates above, and therefore the proof goes along the exact same lines as in the non-compactly supported case. $\square$

5.2. **Besov-Lipschitz boundedness for the high frequency portion of FIOs.** In this section we prove the boundedness of FIOs, where the amplitudes are frequency-supported outside the origin. To this end we have the following:

Proposition 5.4. *Let* $0 < p, q \leqslant \infty$, $m_c(p)$ *as in* (5), $a \in S^m(\mathbb{R}^n)$ *and* $\varphi(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$, *be positively homogeneous of degree one in* ξ. *Then if* $\varphi \in \Phi^2$ *that satisfies the* SND *condition* (8). *Then the operator given by*

$$T_a^\varphi f(x) := \int_{\mathbb{R}^n} (1 - \psi_0(\xi))\, a(x,\xi)\, e^{i\varphi(x,\xi)}\, \widehat{f}(\xi)\, đ\xi, \tag{39}$$

satisfies $T_a^\varphi : B^{s+m-m_c(p)}_{p,q}(\mathbb{R}^n) \to B^s_{p,q}(\mathbb{R}^n)$, *for any* $s \in \mathbb{R}$. *Furthermore, if one assumes that the amplitude* $a(x,\xi)$ *is compactly supported in* x, *then one has the same result, if the phase function* φ *is assumed to be non-degenerate on the support of* $a(x,\xi)$.

Proof. We divide the proof into three steps. In Step 1 we invoke a composition formula which yields a sum of two terms (a main term and a rest term) that need to be analysed separately, and conclude that the main term is L^p bounded (in the sense of Proposition 5.1). In Step 2 we show $B^s_{p,q} \to L^p$ boundedness for the rest term and in Step 3 we complete the proof by deducing the $B^{s+m-m_c(p)}_{p,q} \to B^s_{p,q}$ boundedness.

Step 1 – a composition formula and boundedness of the main term
In the definition of the Besov-Lipschitz norm, the expression $\psi_j(D)T_a^\varphi f$ plays a central role. To obtain favourable estimates for $\psi_j(D)T_a^\varphi f$ we shall use the parameter-dependent composition formula in Theorem 2.13. According to that formula, for any integer $M \geqslant 1$ we can write

$$\psi\left(2^{-j}D\right) T_a^\varphi = \sum_{|\alpha| \leqslant M-1} \frac{2^{-j|\alpha|}}{\alpha!} T_{\sigma_\alpha} + 2^{-jM\varepsilon} T_r =: I + II, \tag{40}$$

where $0 < \varepsilon < \dfrac{1}{2}$. Observe that we have replaced t by 2^{-j} in Theorem 2.13. Now

$$\begin{aligned} &\left|\partial_\xi^\gamma \partial_x^\beta \sigma_\alpha(j,x,\xi)\right| \lesssim 2^{-j|\alpha|(\varepsilon-1)} \langle\xi\rangle^{m-|\alpha|\left(\frac{1}{2}-\varepsilon\right)}, \\ &\operatorname{supp}_\xi \sigma_\alpha(j,x,\xi) = \left\{\xi \in \mathbb{R}^n :\ C_1 2^j \leqslant |\xi| \leqslant C_2 2^j\right\} \text{ and} \\ &\qquad r(j,x,\xi) \in S^{m-M\left(\frac{1}{2}-\varepsilon\right)}(\mathbb{R}^n), \end{aligned}$$

where should mention in passing that $r(j,x,\xi)$ vanishes in a neighborhood of $\xi = 0$. From Proposition 5.1 we have, after a change of variables, that

$$\|T_{\sigma_\alpha} f\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j(m-m_c(p))}\, \|\Psi_j(D) f\|_{L^p(\mathbb{R}^n)}\,. \tag{41}$$

Note that the j-dependence of σ_α is hidden in the notation.

Step 2 – The rest term
We decompose T_r of (40) into Littlewood-Paley pieces as follows:

$$T_r f(x) = \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)}\, r(j,x,\xi)\, \psi_k(\xi)\, \widehat{f}(\xi)\, đ\xi =: \sum_{k=0}^{\infty} T_{r_k} f(x),$$

where the ψ_k's are defined in Definition 2.1. We use the fact that for $0 < p \leqslant \infty$,

$$\|f + g\|_{L^p(\mathbb{R}^n)} \leqslant 2^{C_p} \left(\|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)} \right), \tag{42}$$

where $C_p := \max\left(0, \dfrac{1}{p} - 1\right)$. Now Fatou's lemma and iteration of (42) yield that

$$\begin{aligned} \|T_r f\|_{L^p(\mathbb{R}^n)} &= \left\| \sum_{k=0}^{\infty} T_{r_k} f \right\|_{L^p(\mathbb{R}^n)} \leqslant \liminf_{N\to\infty} \left\| \sum_{k=0}^{N} T_{r_k} f \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \liminf_{N\to\infty} \sum_{k=0}^{N} 2^{kC_p} \|T_{r_k} f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{k=0}^{\infty} 2^{kC_p} \|T_{r_k} f\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

where the hidden constant in the last estimate only depend on p. Therefore, applying Proposition 5.1 with $m - M\left(\dfrac{1}{2} - \varepsilon\right)$ instead of m (recall that r vanishes in a neighborhood of $\xi = 0$), we get that

$$\begin{aligned} \|T_r f\|_{L^p(\mathbb{R}^n)} &\lesssim \sum_{k=0}^{\infty} 2^{kC_p} \|T_{r_k} f\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \sum_{k=0}^{\infty} 2^{k\left(C_p + m - m_c(p) - M\left(\frac{1}{2} - \varepsilon\right)\right)} \|\Psi_k(D) f\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{43}$$

Note that the estimate (43) is uniform in j. Now take

$$M > \max\left(\frac{C_p - s}{\frac{1}{2} - \varepsilon}, \frac{s}{\varepsilon} \right). \tag{44}$$

Then we claim that

$$T_r : B^s_{p,q}(\mathbb{R}^n) \to L^p(\mathbb{R}^n). \tag{45}$$

To see this, we shall analyse the cases $0 < q < 1$ and $1 \leqslant q \leqslant \infty$ separately. Starting with the former, we have

$$\begin{aligned} \|T_r f\|_{L^p(\mathbb{R}^n)} &\lesssim \sum_{k=0}^{\infty} 2^{k\left(C_p + m - m_c(p) - M\left(\frac{1}{2} - \varepsilon\right)\right)} \|\Psi_k(D) f\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \sum_{k=0}^{\infty} 2^{k(s + m - m_c(p))} \|\Psi_k(D) f\|_{L^p(\mathbb{R}^n)} \\ &\leqslant \left(\sum_{k=0}^{\infty} 2^{kq(s + m - m_c(p))} \|\Psi_k(D) f\|^q_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{q}} = \|f\|_{B^{s+m-m_c(p)}_{p,q}(\mathbb{R}^n)}, \end{aligned}$$

where we used (43) for the first inequality and (44) for the second. For $1 \leqslant q \leqslant \infty$ we have in a similar way

$$\begin{aligned}
\|T_r f\|_{L^p(\mathbb{R}^n)} &\lesssim \sum_{k=0}^{\infty} 2^{k\left(C_p+m-m_c(p)-M\left(\frac{1}{2}-\varepsilon\right)\right)} \|\Psi_k(D)f\|_{L^p(\mathbb{R}^n)} \\
&= \sum_{k=0}^{\infty} 2^{k\left(-s+C_p-M\left(\frac{1}{2}-\varepsilon\right)\right)} \left(2^{k(s+m-m_c(p))} \|\Psi_k(D)f\|_{L^p(\mathbb{R}^n)}\right) \\
&\lesssim \left(\sum_{k=0}^{\infty} 2^{kq'\left(-s+C_p-M\left(\frac{1}{2}-\varepsilon\right)\right)}\right)^{\frac{1}{q'}} \left(\sum_{k=0}^{\infty} 2^{kq(s+m-m_c(p))} \|\Psi_k(D)f\|^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&\lesssim \|f\|_{B^{s+m-m_c(p)}_{p,q}(\mathbb{R}^n)}
\end{aligned}$$

and the claim (45) is proven. Note that the calculation above also holds for $q = \infty$ with the usual interpretation of Hölder's inequality.

Step 3 – The $\mathbf{B^{s+m-m_c(p)}_{p,q}} \to \mathbf{B^s_{p,q}}$ boundedness

The results in (41) and (45) yield that

$$\begin{aligned}
\|T^{\varphi}_a f\|_{B^s_{p,q}(\mathbb{R}^n)} &= \left(\sum_{j=0}^{\infty} \left(2^{js} \left\|\psi\left(2^{-j}D\right) T^{\varphi}_a f\right\|_{L^p(\mathbb{R}^n)}\right)^q\right)^{\frac{1}{q}} \\
&\lesssim \left(\sum_{j=0}^{\infty} \left(\sum_{|\alpha|\leqslant M-1} 2^{js} \|T_{\sigma_\alpha} f\|_{L^p(\mathbb{R}^n)} + 2^{-j(M\varepsilon-s)} \|T_r f\|_{L^p(\mathbb{R}^n)}\right)^q\right)^{\frac{1}{q}} \\
&\lesssim \left(\sum_{j=0}^{\infty} \left(2^{j(s+m-m_c(p))} \|\Psi_j(D)f\|_{L^p(\mathbb{R}^n)} + 2^{-j(M\varepsilon-s)} \|f\|_{B^{s+m-m_c(p)}_{p,q}(\mathbb{R}^n)}\right)^q\right)^{\frac{1}{q}} \\
&\lesssim \left(\sum_{j=0}^{\infty} 2^{jq(s+m-m_c(p))} \|\Psi_j(D)f\|^q_{L^p(\mathbb{R}^n)} + \sum_{j=0}^{\infty} 2^{-jq(M\varepsilon-s)} \|f\|^q_{B^{s+m-m_c(p)}_{p,q}(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&\lesssim \left(\|f\|^q_{B^{s+m-m_c(p)}_{p,q}(\mathbb{R}^n)}\right)^{\frac{1}{q}} = \|f\|_{B^{s+m-m_c(p)}_{p,q}(\mathbb{R}^n)},
\end{aligned}$$

and the proof is complete. □

5.3. **Besov-Lipschitz boundedness of the low frequency portion of FIOs.** In this section we prove the boundedness of FIOs, where the amplitudes are frequency-supported in a neighbourhood of the origin. In this case, we will need to distinguish between two cases. First we assume that the amplitude of our FIO is compactly supported in the x-variable. This extra assumption enables us to prove the boundedness for the whole range $0 < p \leqslant \infty$. In the second case, we remove the assumption of compact support in the spatial variable on the amplitude. In this case it turns out that we have to confine ourselves to the range $\dfrac{n}{n+1} < p \leqslant \infty$. We start with the local result. In what follows we set

$$T^{\varphi}_{a_0} f(x) := \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi)\psi_0(\xi)\widehat{f}(\xi)\,đ\xi,$$

where ψ_0 is as in Definition 2.1.

Proposition 5.5 (Local boundedness)**.** *Let $a(x,\xi) \in S^m(\mathbb{R}^n)$ be compactly supported in the x variable and let $\varphi(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$, be positively homogeneous of degree one in ξ, and non-degenerate on the support of $a(x,\xi)$. Then $T^\varphi_{a_0} : B^{s_1}_{p,q_1}(\mathbb{R}^n) \to B^{s_2}_{p,q_2}(\mathbb{R}^n)$, for any $s_1, s_2 \in (-\infty,\infty)$, and $p, q_1, q_2 \in (0,\infty]$.*

Proof. Without loss of generality we can assume that $f = \chi(D)f$ where χ is a smooth cut-off function that is equal to one on the support of ψ_0. Define the self-adjoint operators

$$L_\xi := 1 - \Delta_\xi \text{ and } L_y := 1 - \Delta_y,$$

and note that

$$\langle\xi\rangle^{-2} L_y\, e^{i(x-y)\cdot\xi} = \langle x-y\rangle^{-2} L_\xi\, e^{i(x-y)\cdot\xi} = e^{i(x-y)\cdot\xi}$$

Take integers $N_1 > \dfrac{s_2+n}{2}$ and $N_2 > \dfrac{n}{2p}$. Integrating by parts, we have

$$\begin{aligned}\psi_j(D)T^\varphi_{a_0}f(x) &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{i(x-y)\cdot\xi}\psi_j(\xi)T^\varphi_{a_0}f(y)\,\mathrm{d}y\,\text{đ}\xi \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \langle\xi\rangle^{-2N_1} L_y^{N_1}\Big(\langle x-y\rangle^{-2N_2} L_\xi^{N_2} e^{i(x-y)\cdot\xi}\Big)\psi_j(\xi)T^\varphi_{a_0}f(y)\,\mathrm{d}y\,\text{đ}\xi \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} L_\xi^{N_2}\Big(\langle\xi\rangle^{-2N_1}\psi_j(\xi)\Big)\langle x-y\rangle^{-2N_2} L_y^{N_1}T^\varphi_{a_0}f(y)\,\mathrm{d}y\,\text{đ}\xi.\end{aligned}\tag{46}$$

Since ψ_j is supported on an annulus of size 2^j one has

$$\begin{aligned}\int_{\mathbb{R}^n}\Big|L_\xi^{N_2}\langle\xi\rangle^{-2N_1}\psi_j(\xi)\Big|\,\text{đ}\xi &\lesssim \sum_{|\alpha|\leqslant 2N_2}\int_{|\xi|\sim 2^j}\Big|\partial_\xi^\alpha\Big(\langle\xi\rangle^{-2N_1}\psi_j(\xi)\Big)\Big|\,\text{đ}\xi \\ &\lesssim 2^{jn}\sum_{|\alpha|\leqslant 2N_2}2^{-j(2N_1+|\alpha|)} \lesssim 2^{j(n-2N_1)}\end{aligned}\tag{47}$$

Also, applying Leibniz's and Faà di Bruno's formulae we have that

$$\begin{aligned}T^\varphi_\sigma f(y) := L_y^{N_1}T^\varphi_{a_0}f(y) &= \int_{\mathbb{R}^n} L_y^{N_1}\Big(a(y,\eta)e^{i\varphi(y,\eta)}\Big)\psi_0(\eta)\,\widehat{f}(\eta)\,\text{đ}\eta \\ &= \int_{\mathbb{R}^n}\sigma(y,\eta)\,e^{i\varphi(y,\eta)}\,\widehat{f}(\eta)\,\text{đ}\eta,\end{aligned}\tag{48}$$

with

$$\sigma(y,\eta) := \sum_{|\alpha|\leqslant 2N_1}\sum_{1\leqslant|\beta|\leqslant 2N_1}\sum_{l\leqslant N_1} C_{\alpha,\beta,l}\Big(\partial_y^\alpha a(y,\eta)\Big)\Big(\partial_y^\beta\varphi(y,\eta)\Big)^l\psi_0(\eta).$$

Observe that the assumption on the phase and the mean-value theorem yield $\big|\partial_y^\beta\varphi(y,\eta)\big| = \big|\partial_y^\beta\varphi(y,\eta) - \partial_y^\beta\varphi(y,0)\big| \lesssim |\eta|$, for $|\eta| \neq 0$ and $|\beta| \geqslant 1$. Thus T^φ_σ is the same type of FIO as T^φ_a, and we have

$$\big|\psi_j(D)T^\varphi_{a_0}f(x)\big| \lesssim 2^{j(n-2N_1)}\Big(\langle\cdot\rangle^{-2N_2} * |T^\varphi_\sigma f|\Big)(x).\tag{49}$$

Now using Lemma 2.14 with $b(x,\xi) = \sigma(x,\xi)\,e^{i\varphi(x,\xi)-ix\cdot\xi}$, we can see that the kernel of T^φ_σ satisfies the estimate

$$|K(x,y)| \lesssim \langle x-y\rangle^{-n-\varepsilon}.$$

Therefore, since f is frequency localised, an application of Lemma 2.16 and the fact that σ is compactly supported yield the pointwise estimate

$$|T_\sigma^\varphi f(y)| \lesssim \chi_{\mathcal{K}}(y) \Big(M\left(|f|^r\right)\Big)^{\frac{1}{r}}(y), \tag{50}$$

for $r > \dfrac{n}{n+1}$, where $\mathcal{K} = \operatorname{supp}_y \sigma(y,\xi)$.

Hence, (49), (50) and Peetre's inequality yield

$$\begin{aligned} \left|\psi_j(D)T_{a_0}^\varphi f(x)\right| &\lesssim 2^{j(n-2N_1)} \left(\langle\cdot\rangle^{-2N_2} * \chi_{\mathcal{K}}\Big(M\left(|f|^r\right)\Big)^{\frac{1}{r}}\right)(x) \\ &\lesssim 2^{j(n-2N_1)}\langle x\rangle^{-2N_2} \int_{\mathcal{K}} \Big(M\left(|f|^r\right)\Big)^{\frac{1}{r}}(y)\, \mathrm{d}y. \end{aligned} \tag{51}$$

Now taking the L^p norm, choosing N_2 large enough, using the L^∞ boundedness of the Hardy-Littlewood maximal operator, and finally using Lemma 2.19, we obtain for $0 < p \leqslant \infty$

$$\begin{aligned} \|\psi_j(D)T_{a_0}^\varphi f(x)\|_{L^p(\mathbb{R}^n)} &\lesssim 2^{j(n-2N_1)}\||f|^r\|_{L^\infty(\mathbb{R}^n)}^{\frac{1}{r}} \lesssim 2^{j(n-2N_1)} \|f\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim 2^{j(n-2N_1)} \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{52}$$

Thus (52) yields that

$$\begin{aligned} \left\|T_{a_0}^\varphi f\right\|_{B^{s_2}_{p,q_2}(\mathbb{R}^n)} &= \left(\sum_{j=0}^\infty 2^{js_2q_2} \|\psi_j(D)T_a^\varphi f\|^{q_2}_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q_2}} \\ &\lesssim \left(\sum_{j=0}^\infty 2^{jq_2(s_2+n-2N_1)} \|f\|^{q_2}_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q_2}} \\ &= \|f\|_{L^p(\mathbb{R}^n)} \left(\sum_{j=0}^\infty 2^{jq_2(s_2+n-2N_1)}\right)^{\frac{1}{q_2}} \\ &\lesssim \|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{B^{s_1}_{p,q_1}(\mathbb{R}^n)}. \end{aligned}$$

□

Now we state and prove the global boundedness of FIOs with frequency localised amplitudes on Besov-Lipschitz spaces.

Proposition 5.6 (Global boundedness)**.** *Let* $a(x,\xi) \in S^m(\mathbb{R}^n)$ *and* $\varphi(x,\xi) \in \Phi^2$ *and verifies the* SND *condition. Then* $T_{a_0}^\varphi : B^{s_1}_{p,q_1}(\mathbb{R}^n) \to B^{s_2}_{p,q_2}(\mathbb{R}^n)$*, for any* $s_1, s_2 \in (-\infty,\infty)$*,* $q_1, q_2 \in (0,\infty]$ *and* $p \in \left(\dfrac{n}{n+1}, \infty\right]$.

Proof. First we use Lemma 2.15 to reduce the operator to finite sums of operators of the form

$$\int_{\mathbb{R}^n} a_0(x,\xi)\, e^{i\theta(x,\xi)+i\nabla_\xi\varphi(x,\zeta)\cdot\xi}\, \widehat{u}(\xi)\, \text{đ}\xi$$

where ζ is a point on the unit sphere $\mathbb{S}^{n-1}$, $\theta(x,\xi) \in \Phi^1$, and $a_0(x,\xi) \in S^m(\mathbb{R}^n)$ is localised in the ξ variable around the point ζ. Then observe that if $t(x) = \nabla_\xi\varphi(x,\zeta)$,

then due to the SND condition on the phase, $t(x)$ is a global diffeomorphism and the Jacobian matrix of $t(x)$, $Dt(x) = \Big(\partial^2_{x_j\xi_k}\varphi(x,\zeta)\Big)$, has bounded entries (by the $\varphi \in \Phi^2$ assumption) and hence $|\det Dt(x)| \lesssim 1$.
This enables us to use the invariance of Besov-Lipschitz spaces under diffeomorphisms (Theorem 2.5) to reduce the proof of the proposition, to the case of operators of the form

$$T^\varphi_{a_0} f(x) = \int_{\mathbb{R}^n} a_0(x,\xi)\, e^{i\varphi(x,\xi)}\, \widehat{u}(\xi)\; đ\xi,$$

where $a_0 \in S^m$ and $\varphi(x,\xi) = \theta(x,\xi) + x\cdot\xi$ with $\theta \in \Phi^1$.

Now the rest of the proof differs only marginally from that of Proposition 5.5. First we once again without loss of generality assume that $f = \chi(D)f$ where χ is a smooth cut-off function that is equal to one on the support of ψ_0. Then considering $\psi_j(D)T^\varphi_{a_0} f(x)$ as an oscillatory integral, we can deduce that the integral representation (46) is valid for $\psi_j(D)T^\varphi_{a_0}$ even in the current case. Then once again using Lemma 2.14 with $b(x,\xi) = \sigma(x,\xi)\, e^{i\theta(x,\xi)}$ (σ is as in Proposition 5.5) and the fact that $\theta \in \Phi^1$, we can see that the kernel of T^φ_σ satisfies the estimate

$$|K(x,y)| \lesssim \langle x-y\rangle^{-n-\varepsilon}.$$

Moreover, from Lemma 2.16, it follows that for $r > \dfrac{n}{n+1}$

$$\begin{aligned}\left|\psi_j(D)T^\varphi_{a_0} f(x)\right| &\lesssim 2^{j(n-2N_1)} \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \langle x-z\rangle^{-2N_2}\langle z-y\rangle^{-n-\varepsilon}\,\mathrm{d}z\right)|f(y)|\,\mathrm{d}y\\ &\lesssim 2^{j(n-2N_1)}\int_{\mathbb{R}^n}\langle x-y\rangle^{-n-\varepsilon}\,|f(y)|\,\mathrm{d}y \lesssim 2^{j(n-2N_1)}\Big(M\left(|f|^r\right)\Big)^{\frac{1}{r}}(x).\end{aligned}$$

This yields that for $r < p \leqslant \infty$ one has

$$\|\psi_j(D)T^\varphi_{a_0} f(x)\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j(n-2N_1)}\,\|f\|_{L^p(\mathbb{R}^n)}\,.$$

and the proof can be concluded following the same argument as in the proof of Proposition 5.5. □

5.4. **Local and Global boundedness of FIOs on Besov-Lipschitz spaces.** In this section we state and prove the local and global boundedness of Fourier integral operators on Besov-Lipschitz spaces. In light of the results of the previous sections, what remains to do is to basically put all the bits and pieces (i.e. the high and low frequency results for various cases) together. As usual, we set

$$T^\varphi_a f(x) := \int_{\mathbb{R}^n} a(x,\xi)\, e^{i\varphi(x,\xi)}\,\widehat{f}(\xi)\, đ\xi.$$

Our main local and global boundedness results are

Theorem 5.7. *Let $a(x,\xi) \in S^m(\mathbb{R}^n)$, $p \in (0,\infty]$ and $m_c(p) := -(n-1)\left|\dfrac{1}{p}-\dfrac{1}{2}\right|$. Assume also that $\varphi(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n\setminus\{0\})$, is positively homogeneous of degree one in ξ. Then under these assumptions, the following results hold true*:

(i) *If $a(x,\xi)$ has compact support in x and $\varphi(x,\xi)$ is non-degenerate on the support of $a(x,\xi)$, then for any $s\in\mathbb{R}$, $0<p\leqslant\infty$ and $0<q\leqslant\infty$*

$$T_a^\varphi : B^{s+m-m_c(p)}_{p,q}(\mathbb{R}^n)\to B^s_{p,q}(\mathbb{R}^n),$$

(ii) *If $\varphi(x,\xi)\in\Phi^2$ is SND, then for any $s\in\mathbb{R}$, $\dfrac{n}{n+1}<p\leqslant\infty$ and $0<q\leqslant\infty$*

$$T_a^\varphi : B^{s+m-m_c(p)}_{p,q}(\mathbb{R}^n)\to B^s_{p,q}(\mathbb{R}^n).$$

In particular taking $m=m_c(p)$ in both cases, we have that

$$T_a^\varphi : B^{s}_{p,q}(\mathbb{R}^n)\to B^s_{p,q}(\mathbb{R}^n).$$

Proof. Once again we split T_a^φ into a low and a high frequency part. Indeed, take ψ_0 as in Definition 2.1, i.e.

$$\begin{aligned}T_a^\varphi f(x) &= \int_{\mathbb{R}^n}\psi_0(\xi)a(x,\xi)e^{i\varphi(x,\xi)}\widehat{f}(\xi)\,\text{đ}\xi\\ &\quad+\int_{\mathbb{R}^n}(1-\psi_0(\xi))a(x,\xi)e^{i\varphi(x,\xi)}\widehat{f}(\xi)\,\text{đ}\xi\\ &=: T_1f(x)+T_2f(x).\end{aligned}$$

Now for (i) we use Proposition 5.5 and for (ii) Proposition 5.6 (taking $s_1=s+m-m_c(p)$, $s_2=s$ and $q_1=q_2=q$). These yield that $T_1 : B^{s+m-m_c(p)}_{p,q}(\mathbb{R}^n)\to B^s_{p,q}(\mathbb{R}^n)$. For T_2f Proposition 5.4 yields that $T_2 : B^{s+m-m_c(p)}_{p,q}(\mathbb{R}^n)\to B^s_{p,q}(\mathbb{R}^n)$. □

6. Boundedness of FIOs on Triebel-Lizorkin spaces

In this section we investigate the boundedness of FIO's on Triebel-Lizorkin spaces spaces $F^s_{p,q}(\mathbb{R}^n)$ for $-\infty<s<\infty$, $0<p\leqslant\infty$ and $0<q\leqslant\infty$. Some of the results that we drive are based on the Besov-Lipschitz results which were obtained in the previous sections, a couple are obtained by interpolation, and some through direct methods. Once again, both local and global cases will be treated here. We start with the following result which is sharp, up to the end point.

Theorem 6.1. *Let $a(x,\xi)\in S^m(\mathbb{R}^n)$, $p\in(0,\infty]$ and $m_c(p):=-(n-1)\left|\dfrac{1}{p}-\dfrac{1}{2}\right|$. Assume also that $\varphi(x,\xi)\in\mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$, is positively homogeneous of degree one in ξ. If $m<m_c(p)$, then in either of the following cases, we have that T_a^φ is bounded from $F^s_{p,q}(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$.*

(i) *$s\in\mathbb{R}$, $0<p<\infty$ and $0<q\leqslant\infty$; $a(x,\xi)$ has compact support in x, and φ is non-degenerate on the support of $a(x,\xi)$,*
(ii) *$s\in\mathbb{R}$, $\dfrac{n}{n+1}<p<\infty$, $0<q\leqslant\infty$, $\varphi(x,\xi)\in\Phi^2$ is SND.*

Proof. Take $\varepsilon>0$ in such a way that $m+2\varepsilon\leqslant m_c$. Then using the embedding (4), equality (3), Theorem 5.7, and finally the fact that $(1-\Delta)^{-\frac{\varepsilon}{2}}$ is an isomorphism from $F^s_{p,q}(\mathbb{R}^n)$ to $F^{s+\varepsilon}_{p,q}(\mathbb{R}^n)$, we have that

$$\begin{aligned}\|Tf\|_{F^s_{p,q}(\mathbb{R}^n)} &= \|T(1-\Delta)^{\varepsilon}(1-\Delta)^{-\varepsilon}f\|_{F^s_{p,q}(\mathbb{R}^n)}\\ &\leqslant \|T(1-\Delta)^{\varepsilon}(1-\Delta)^{-\varepsilon}f\|_{F^{s+\varepsilon}_{p,p}(\mathbb{R}^n)} \lesssim \|(1-\Delta)^{-\varepsilon}f\|_{F^{s+\varepsilon}_{p,p}(\mathbb{R}^n)}\\ &\leqslant \|(1-\Delta)^{-\varepsilon}f\|_{F^{s+2\varepsilon}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^s_{p,q}(\mathbb{R}^n)}.\end{aligned}$$

□

But indeed this result can be extended to the endpoint $m = m_c$ if $q = 2$, at least for $0 < p \leqslant 2$. This, in the local case, i.e. the case of amplitudes with compact spatial support p could be taken in the interval $(0,\infty)$. However, with the conditions of Theorem 6.1 above, one can prove a global version of the boundedness of FIOs on $F^s_{p,2}$, whose proof is based on the techniques developed by Seeger-Sogge-Stein [20] and M. Ruzhansky and M. Sugimoto [18]. The long and rather technical proof will occupy the next subsection.

6.1. Triebel-Lizorkin boundedness of the high frequency portion of FIOs.

First we consider the boundedness of FIOs with high frequency amplitudes on Triebel-Lizorkin spaces $F^0_{p,2}(\mathbb{R}^n)$ for $0 < p \leqslant 1$. As was mentioned in Definition 2.4, $F^0_{p,2}(\mathbb{R}^n) = h^p(\mathbb{R}^n)$ is the local Hardy space of Goldberg's [4], and we shall to use the atomic decomposition of these spaces in order to carry out our agenda.

Proposition 6.2. *Let ψ_0 be as cut-off function as in* Definition 2.1*, $p \in (0,\infty]$ and $m_c(p)$ the critical order defined in* (5)*. Assume that $a \in S^{m_c(p)}(\mathbb{R}^n)$ and $\varphi \in \Phi^2$ is a phase function that verifies the* SND *condition* (8)*. Then for $n \geqslant 2$ the operator given by*

$$T^{\varphi}_a f(x) := \int_{\mathbb{R}^n} (1-\psi_0(\xi))\, a(x,\xi)\, e^{i\varphi(x,\xi)}\, \widehat{f}(\xi)\, đ\xi,$$

satisfies $F^s_{p,2}(\mathbb{R}^n) \to F^s_{p,2}(\mathbb{R}^n)$, for $-\infty < s < \infty$.

Proof. Set $\sigma(x,\xi) := (1-\psi_0(\xi))\, a(x,\xi)$. We divide the proof in different steps as follows.

(i) In Step 1 we consider the case when $s = 0$, $0 < p \leqslant 1$, $n \geqslant 2$ and an atom a with support inside a ball with radius $r \leqslant 1$. We show that for $T := T^{\varphi}_{\sigma}$ or $T := (T^{\varphi}_{\sigma})^*$, $\|Ta\|_{L^p(\mathbb{R}^n)} \leqslant C$, where the constant C doesn't depend of a and r.

(ii) In Step 2 we assume the same premises as in Step 1 with the only difference that $r \geqslant 1$. Step 1 and 2 will together imply that $T : h^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$.

(iii) In Step 3 we lift the result to $T : h^p(\mathbb{R}^n) \to h^p(\mathbb{R}^n)$.

(iv) We conclude the proof by showing the boundedness of T on $F^s_{p,2}(\mathbb{R}^n)$ for $s \in \mathbb{R}$, $0 < p \leqslant 1$. A duality argument then yields that any FIO with the corresponding critical decay, is bounded on $F^s_{\infty,2}(\mathbb{R}^n)$ for $s \in \mathbb{R}$. In particular, for $s = 0$, we obtain the global bmo boundedness of FIOs. Finally, using interpolation we obtain the boundedness of FIOs on $F^s_{p,2}(\mathbb{R}^n)$ for the whole range $0 < p \leqslant \infty$.

These steps will conclude the proof.

Step 1 – Estimates of the $\mathbf{L^p}$ norm when $\mathbf{r \leqslant 1}$
Let $a(x)$ be an h^p-atom supported in a ball B of radius $r \leqslant 1$. Now split

$$\int_{\mathbb{R}^n} |Ta(x)|^p \,\mathrm{d}x = \int_{B^*} |Ta(x)|^p \,\mathrm{d}x + \int_{B^{*c}} |Ta(x)|^p \,\mathrm{d}x, \tag{53}$$

where B^* is defined in (24). Hölder's inequality yields that

$$\int_{B^*} |Ta(x)|^p \,\mathrm{d}x \leqslant |B^*|^{1-\frac{p}{2}} \left(\int_{B^*} |Ta(x)|^2 \,\mathrm{d}x \right)^{\frac{p}{2}} \lesssim r^{\left(1-\frac{p}{2}\right)} \|Ta\|_{L^2}^p.$$

To analyse the second term on the right hand side of (53) we proceed as follows.

First assume that $-\dfrac{n}{2} < m < 0$. Then there exists a $1 < q < 2$ such that $\dfrac{1}{2} = \dfrac{1}{q} + \dfrac{m}{n}$. Observe that it is at this point where the assumption on the dimension n plays a role. Indeed the case $n = 1$ cannot satisfy this assumption, as then $m = 0$. Using the global $L^2 \to L^2$ boundedness of the operator T and the estimates for Riesz potentials we can deduce that T is bounded from $L^q(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and therefore

$$\|Ta\|_{L^2(\mathbb{R}^n)}^p \lesssim \|a\|_{L^q(\mathbb{R}^n)}^p \lesssim c|B|^{\frac{p}{q}-1} \lesssim r^{n\left(\frac{p}{q}-1\right)}.$$

Thus

$$\int_{B^*} |Ta(x)|^p \,\mathrm{d}x \lesssim r^{1-\frac{p}{2}+n\left(\frac{p}{q}-1\right)}.$$

To see that $r^{1-\frac{p}{2}+n\left(\frac{p}{q}-1\right)} \lesssim 1$, we observe that since $\dfrac{p}{q} = \dfrac{p}{2} - \dfrac{pm}{n}$ we have

$$\begin{aligned} 1 - \frac{p}{2} + n\left(\frac{p}{q} - 1\right) &= 1 - \frac{p}{2} + n\left(\frac{p}{2} - \frac{pm}{n} - 1\right) = p\left(\frac{1}{p} - \frac{1}{2} + \frac{n}{2} - m - \frac{n}{p}\right) \\ &= p\left(-(n-1)\left(\frac{1}{p} - \frac{1}{2}\right) - m\right). \end{aligned}$$

Now since $m \leqslant -(n-1)\left(\dfrac{1}{p} - \dfrac{1}{2}\right)$, it follows that $r^{1-\frac{p}{2}+n\left(\frac{p}{q}-1\right)} \lesssim 1$.

If instead $m \leqslant -\dfrac{n}{2}$, then setting $b = |B|^{\frac{1}{p}-\frac{1}{q}}a$, with $\dfrac{1}{2} = \dfrac{1}{q} + \dfrac{m}{n}$ as before (so now $q < p < 1$) we see that b is an h^q-atom with the same support as a. In fact b becomes an atom in the Hardy space $H^q(\mathbb{R}^n)$, so by the results in [8, Corollary 2.3], we have that $T : H^q(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bounded and

$$\begin{aligned} \int_{B^*} |Ta(x)|^p \,\mathrm{d}x &\lesssim r^{1-\frac{p}{2}} \|a\|_{H^q}^p \lesssim r^{1-\frac{p}{2}} |B|^{p\left(\frac{1}{q}-\frac{1}{p}\right)} \|b\|_{H^q}^p \\ &\lesssim r^{1-\frac{p}{2}+n\left(\frac{p}{q}-1\right)} \lesssim 1. \end{aligned}$$

Using the partition of unity that was introduced in (16) we can write

$$T = \sum_{j=0}^{+\infty} T_j =: \sum_j \sum_\nu T_j^\nu. \tag{54}$$

Now to deal with the integral $\int_{B^{*c}} |Ta(x)|^p \, \mathrm{d}x$, using the notation in (54), we observe that

$$\int_{B^{*c}} |Ta(x)|^p \, \mathrm{d}x \leqslant \sum_{2^j \geqslant r^{-1}} \int_{B^{*c}} |T_j a(x)|^p \, \mathrm{d}x + \sum_{2^j < r^{-1}} \int_{B^{*c}} |T_j a(x)|^p \, \mathrm{d}x. \tag{55}$$

Therefore, Lemma 3.4 part (iii) yields that

$$\begin{aligned}\sum_{2^j \geqslant r^{-1}} \int_{B^{*c}} |T_j a(x)|^p \, \mathrm{d}x &\leqslant \sum_{2^j \geqslant r^{-1}} \int_{B^{*c}} \sup_{y\in B} |K_j(x,y)|^p \, \mathrm{d}x \left(\int_B |a(y)| \mathrm{d}y\right)^p \\ &\lesssim \sum_{2^j \geqslant r^{-1}} 2^{jn(p-1)} \, 2^{-j} \, r^{-1} \left(\int_B |a(y)| \mathrm{d}y\right)^p \\ &\lesssim \sum_{2^j \geqslant r^{-1}} 2^{jn(p-1)} \frac{2^{-j}}{r} r^{n(p-1)} \lesssim 1\end{aligned}$$

For the second term in (55) we use the moment condition on the atom a, which holds since $r \leqslant 1$. The kernel estimate (ii) above and the fact that $M = \left[n\left(\frac{1}{p} - 1\right)\right] + 1$ now yield

$$\begin{aligned}&\sum_{2^j < r^{-1}} \int_{B^{*c}} |T_j a(x)|^p \, \mathrm{d}x \\ &= \sum_{2^j < r^{-1}} \int_{B^{*c}} \left| \int_B (K_j(x,y) - p_j(x,\ y - \overline{y})) a(y) \, \mathrm{d}y \right|^p \, \mathrm{d}x \\ &\lesssim \sum_{2^j < r^{-1}} \int_{B^{*c}} \sup_{y\in B} |K_j(x,y) - p_j(x,\ y - \overline{y})|^p \left(\int_B |a(y)| \, \mathrm{d}y\right)^p \mathrm{d}x \\ &\lesssim \sum_{2^j < r^{-1}} 2^{jMp} \, 2^{jn(p-1)} \, r^{Mp} \left(\int_B |a(y)| \, \mathrm{d}y\right)^p \\ &\lesssim \sum_{2^j < r^{-1}} 2^{jMp} \, 2^{jn(p-1)} r^{Mp+n(p-1)} \lesssim 1.\end{aligned}$$

This proves (53) for balls of radius less than or equal to one.

Step 2 – Estimates of the $\mathbf{L^p}$ norm when $\mathbf{r > 1}$

To prove that $\int_{\mathbb{R}^n} |Ta(x)|^p \, \mathrm{d}x \lesssim 1$ when the support of the atom a lies inside a ball of radius larger than one, we need to use a different strategy. First we observe that a global norm-estimate for Ta with a supported in a ball with an arbitrary centre, would follow from a uniform in s norm-estimate for $\tau_s^* T \tau_s a$, with an atom a whose support is inside a ball centred at the origin. This because by translation invariance of the h^p norm one has that $\|Ta\|_{h^p(\mathbb{R}^n)} = \|\tau_s^* T \tau_s \tau_{-s} a\|_{h^p(\mathbb{R}^n)}$. Note that here τ_s is the operator of translation by $s \in \mathbb{R}^n$. Thus our goal is to establish that $\|\tau_s^* T \tau_s a\|_{L^p(\mathbb{R}^n)} \lesssim 1$, where the estimate is uniform in s.

At this point we once again use the conditions on the phase function and Theorem 2.5 on the invariance of Triebel-Lizorkin spaces under diffeomorphisms as in the proof of Lemma 5.6, to reduce our analysis to the case of operators with φ of the form $\theta(x,\xi)+x\cdot\xi$, or $-\theta(y,\xi)-iy\cdot\xi$ with $\theta\in\Phi^1$. Now let $r\geqslant 1$, $0<p\leqslant 1$, $L>\dfrac{n}{p}$ and $s\in\mathbb{R}^n$. Suppose a is an h^p atom supported in a ball B, centred at the origin, with radius r. Split the L^p norm of $\tau_s^*T\tau_s a$ into following two pieces:

$$\|\tau_s^*T\tau_s a\|_{L^p(\mathbb{R}^n)}\leqslant\|\tau_s^*T\tau_s a\|_{L^p\left(\widetilde{\Delta}_{2r}\right)}+\|\tau_s^*T_\sigma^\varphi\tau_s a\|_{L^p\left(\mathbb{R}^n\setminus\widetilde{\Delta}_{2r}\right)}$$

First let us show that

$$\|\tau_s^*T\tau_s a\|_{L^p\left(\widetilde{\Delta}_{2r}\right)}\leqslant C(n,M_L,N_{L+1}).$$

For $x\in\widetilde{\Delta}_{2r}$ and $|y|\leqslant r$, we have $\widetilde{H}(x)\leqslant 2H(x,x-y)$ and $(x,x-y)\in\Delta_r$ by Lemma 4.1. This fact and Lemma 4.2 yield

$$\begin{aligned}|Ta(x)|&\leqslant 2^L\widetilde{H}(x)^{-L}\int_{|y|\leqslant r}\left|H(x,x-y)^LK(x,x-y)a(y)\right|\,\mathrm{d}y\\&\leqslant 2^L\widetilde{H}(x)^{-L}\left\|H^LK\right\|_{L^\infty(\Delta_r)}\|a\|_{L^1}\leqslant C(n,L,M_L,N_{L+1})\widetilde{H}(x)^{-L},\end{aligned}$$

since $\|a\|_{L^1}\leqslant|B|^{1-\frac{1}{p}}\lesssim r^{n\left(1-\frac{1}{p}\right)}$, and $r\geqslant 1$. Therefore choosing $L>\dfrac{n}{p}$, Lemma 4.2 and the monotonicity of Δ_r yield

$$\|Ta\|_{L^p\left(\widetilde{\Delta}_{2r}\right)}\leqslant\left\|\widetilde{H}(x)^{-L}\right\|_{L^p\left(\widetilde{\Delta}_{2r}\right)}\leqslant C(n,M_L,N_{L+1}).$$

Now since the phase function and the amplitude of $\tau_s^*T\tau_s$ are either of the form $\theta(x+s,\xi)+(x-y)\cdot\xi$ and $\sigma(x+s,\xi)$, or of the form $-\theta(y+s,\xi)+(x-y)\cdot\xi$ and $\overline{\sigma(y+s,\xi)}$, we see that the conjugation of T by τ_s renders the constants M_L and N_{L+1} unchanged and therefore the estimate above also yields the very same one for $\tau_s^*T\tau_s$. This means that $\|\tau_s^*T\tau_s a\|_{L^p\left(\widetilde{\Delta}_{2r}\right)}\lesssim 1$.

On the other hand for $\|\tau_s^*T\tau_s a\|_{L^p\left(\mathbb{R}^n\setminus\widetilde{\Delta}_{2r}\right)}$, Lemma 4.1, Hölder's inequality and the properties of the atom a yield that

$$\begin{aligned}\|\tau_s^*T\tau_s a\|_{L^p\left(\mathbb{R}^n\setminus\widetilde{\Delta}_{2r}\right)}&\leqslant\left|\mathbb{R}^n\setminus\widetilde{\Delta}_{2r}\right|^{1-\frac{p}{2}}\|\tau_s^*T\tau_s a\|^p_{L^2(\mathbb{R}^n)}\\&\lesssim r^{n\left(1-\frac{p}{2}\right)}\|a\|^p_{L^2(\mathbb{R}^n)}\lesssim r^{n\left(1-\frac{p}{2}\right)}r^{n\left(\frac{p}{2}-1\right)}\lesssim 1.\end{aligned}$$

Step 3 – Lifting the result to $\mathbf{h^p}\to\mathbf{h^p}$ boundedness

In order to boost up this to the desired $h^p\to h^p$ boundedness, we follow the strategy in [14] in order to show that $\mathbf{I}\in h^r(\mathbb{R}^n)$. As was shown in that paper, in order to show that $f\in h^p(\mathbb{R}^n)$ it is enough to prove that $r_\varepsilon^\alpha(D)f\in L^p(\mathbb{R}^n)$ uniformly in ε, where $\varepsilon\in(0,1]$, $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_+^n$, $r_\varepsilon^\alpha(\xi)=\widehat{\Psi}(\varepsilon\xi)\prod_{i=1}^n\left(\dfrac{\xi_i}{|\xi|}\right)^{\alpha_i}\left(1-\widehat{\Theta}(\xi)\right)^{\alpha_i}$, $\Psi\in\mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\displaystyle\int_{\mathbb{R}^n}\Psi(x)\,\mathrm{d}x=1$, and $\widehat{\Theta}$ is the smooth cut-off function which is identically one in a neighborhood of the origin. Moreover

$$\left\|\widehat{\Theta}(D)f\right\|_{L^p(\mathbb{R}^n)}+\sum_{M\leqslant|\alpha|\leqslant M+1}\sup_{0<\varepsilon\leqslant 1}\|r_\varepsilon^\alpha(D)f\|_{L^p(\mathbb{R}^n)}\sim\|f\|_{h^p(\mathbb{R}^n)},$$

where $M=\left[n\left(\frac{1}{p}-1\right)\right]+1$. As a consequence, it will be enough to prove that

$$\|r_\varepsilon^\alpha(D)Tf\|_{L^p(\mathbb{R}^n)}\lesssim\|f\|_{h^p(\mathbb{R}^n)}, \tag{56}$$

uniformly in ε, and

$$\left\|\widehat{\Theta}(D)Tf\right\|_{L^p(\mathbb{R}^n)}\lesssim\|f\|_{h^p(\mathbb{R}^n)}. \tag{57}$$

Suppose first that $T=T_\sigma^\varphi$. If the phase function $\varphi\in\Phi^2$ and is SND, then using our phase reduction mentioned previously, it is not hard to show that for a reduced phase φ, one has $|\nabla_x\varphi(x,\xi)|\sim|\xi|$, and for $|\alpha|$, $|\beta|\geqslant 1$ we have $|\partial_x^\alpha\varphi(x,\xi)|\lesssim\langle\xi\rangle$, $|\partial_\xi^\alpha\partial_x^\beta\varphi(x,\xi)|\lesssim|\xi|^{1-|\alpha|}$ (observe also that $|\xi|$ is large). Moreover $r_\varepsilon^\alpha(D)$ and $\widehat{\Theta}(D)$ are pseudodifferential operators with symbols respectively in $S^0(\mathbb{R}^n)$ (uniformly in ε), and in $S^{-\infty}(\mathbb{R}^n)$. Therefore, using Theorem 2.13 with $t=1$, we can see that the compositions $r_\varepsilon^\alpha(D)T$ and $\widehat{\Theta}(D)T$ are FIOs with amplitudes in $S^{m_c}(\mathbb{R}^n)$ and $S^{-\infty}(\mathbb{R}^n)$ and phase functions φ, and therefore (56) and (57) are both valid.

Assume now that $T=(T_\sigma^\varphi)^*$. Observe that for any real-valued Fourier multiplier $p(\xi)\in S^s(\mathbb{R}^n)$ one has that $P(D)(T_\sigma^\varphi)^*=(T_\sigma^\varphi P(D))^*$. Now $T_\sigma^\varphi P(D)$ is an FIO with the phase φ and an amplitude in $S^{m_c(p)+s}(\mathbb{R}^n)$. In particular, if $P(D)$ is either of r_ε^α or $\widehat{\Theta}$ then $T_\sigma^\varphi P(D)$ is an FIO with phase φ and an amplitude in the class $S^{m_c(p)}(\mathbb{R}^n)$. However, in Step 1 and 2 of this proof, we have shown that the adjoints of such operators are bounded from h^p to L^p and therefore $P(D)(T_\sigma^\varphi)^*$ is also bounded from h^p to L^p and once again (56) and (57) are valid uniformly in $\varepsilon\in(0,1]$. Therefore we have that T is bounded on h^p.

Step 4 – Lifting to $\mathbf{F^s_{p,2}}$
Since T has been proven to be bounded from $F^0_{p,2}(\mathbb{R}^n)$ to itself, we once again use Theorem 2.13 to see that the operator $(1-\Delta)^{\frac{s}{2}}T_a^\varphi(1-\Delta)^{-\frac{s}{2}}$ is a similar operator associated to an amplitude in $S^{m_c}(\mathbb{R}^n)$ and phase φ, and hence bounded from $F^0_{p,2}(\mathbb{R}^n)$ to itself. Therefore using the fact that the operator $(1-\Delta)^{\frac{s}{2}}$ is an isomorphism from $F^s_{p,2}(\mathbb{R}^n)$ to $F^0_{p,2}(\mathbb{R}^n)$ for $0<p\leqslant 1$, we obtain the desired result of Proposition 6.2. Finally, duality and interpolation yields the result for $p\in(0,\infty]$.

□

6.2. **Triebel-Lizorkin boundedness of the low frequency portion of FIOs.** In this section we prove the boundedness of FIOs, where the amplitudes are frequency-supported in a neighbourhood of the origin. This is quite similar to the case of Besov-Lipschitz spaces and we shall use the estimates that were developed in that context. In what follows we set

$$T_{a_0}^\varphi f(x):=\int_{\mathbb{R}^n}e^{i\varphi(x,\xi)}a(x,\xi)\psi_0(\xi)\widehat{f}(\xi)\,\text{đ}\xi$$

where ψ_0 is as in Definition 2.1. We start with the local result:

Proposition 6.3 (Local boundedness)**.** *Let $a(x,\xi)\in S^m(\mathbb{R}^n)$, with compact support in x and $\varphi(x,\xi)\in \mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$ be positively homogeneous of degree one in ξ and non-degenerate on the support of $a(x,\xi)$. Then $T^\varphi_{a_0}: F^{s_1}_{p,q_1}(\mathbb{R}^n)\to F^{s_2}_{p,q_2}(\mathbb{R}^n)$, for $0<p,q_1,q_2\leqslant\infty$, $-\infty<s_1,s_2<\infty$.*

Proof. Using (52), for $0<p\leqslant\infty$ we have the pointwise estimate

$$\left|\psi_j(D)T^\varphi_{a_0}f(x)\right|\lesssim 2^{j(n-2N_1)}\left\|f\right\|_{L^p(\mathbb{R}^n)},$$

from which it follows that

$$\begin{aligned}\left\|T^\varphi_{a_0}f\right\|_{F^{s_2}_{p,q_2}(\mathbb{R}^n)} &= \left\|\left(\sum_{j=0}^\infty 2^{js_2q_2}|\psi_j(D)T^\varphi_{a_0}f|^{q_2}\right)^{\frac{1}{q_2}}\right\|_{L^p(\mathbb{R}^n)}\\ &\lesssim \left(\sum_{j=0}^\infty 2^{jq_2(s_2+n-2N_1)}\left\|f\right\|^{q_2}_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q_2}}\\ &= \left\|f\right\|_{L^p(\mathbb{R}^n)}\left(\sum_{j=0}^\infty 2^{jq_2(s_2+n-2N_1)}\right)^{\frac{1}{q_2}}\\ &\lesssim \left\|f\right\|_{L^p(\mathbb{R}^n)}\lesssim \left\|f\right\|_{F^{s_1}_{p,q_1}(\mathbb{R}^n)}.\end{aligned}$$

□

Now we state and prove the global boundedness of FIOs with frequency localised amplitudes on Triebel-Lizorkin spaces. The proof of this is similar to that of Propositions 5.6, and 6.3 and hence is omitted.

Proposition 6.4 (Global boundedness)**.** *Let $a(x,\xi)\in S^m(\mathbb{R}^n)$ and $\varphi(x,\xi)\in\Phi^2$ verifies the* SND *condition. Then $T^\varphi_{a_0}: F^{s_1}_{p,q_1}(\mathbb{R}^n)\to F^{s_2}_{p,q_2}(\mathbb{R}^n)$, for $\frac{n}{n+1}<p\leqslant\infty$, $0<q_1,q_2\leqslant\infty$, $-\infty<s_1,s_2<\infty$.*

6.3. **Local and Global boundedness of FIOs on Triebel-Lizorkin spaces.** In this section we state and prove the local and global boundedness of Fourier integral operators on Triebel-Lizorkin spaces. In light of the results of the previous sections, what remains to do is to put the high and low frequency results for various cases together.

Our main local and global boundedness results are

Theorem 6.5. *Let $p\in(0,\infty]$, $a(x,\xi)\in S^{m_c(p)}(\mathbb{R}^n)$ and $\varphi(x,\xi)\in\mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$, be positively homogeneous of degree one in ξ. Then under these assumptions, the following results hold true*:

(i) *If $a(x,\xi)$ has compact support in x and φ is non-degenerate on the support of $a(x,\xi)$, then for any $s\in\mathbb{R}$ and $0<p\leqslant\infty$ the operator T^φ_a is bounded from $F^s_{p,2}(\mathbb{R}^n)$ to $F^s_{p,2}(\mathbb{R}^n)$.*

(ii) *If $\varphi(x,\xi)\in\Phi^2$ is* SND*, then for any $s\in\mathbb{R}$ and $\frac{n}{n+1}<p\leqslant\infty$, the operator T^φ_a is bounded from $F^s_{p,2}(\mathbb{R}^n)$ to $F^s_{p,2}(\mathbb{R}^n)$.*

Proof. For the proof of (i), one observes that the compact support, the homogeneity and the non-degeneracy of the phase function yield that

$$\left|\det\left(\partial^2_{x_j\xi_k}\varphi(x,\xi)\right)\right| \geqslant \min_{(x,\xi)\,\in\,\mathrm{supp}\ a\times\mathbb{S}^{n-1}} \left|\det\left(\partial^2_{x_j\xi_k}\varphi(x,\xi)\right)\right| > 0.$$

Moreover, the same conditions on the phase also yield that $\varphi \in \Phi^2$. Thus for the high frequency portion of the FIO, the desired boundedness follows from the same arguments as in the proof of Proposition 6.2. Now adding the low frequency result of Proposition 6.3, we can conclude the proof of (i).

To prove (ii) one just combines the results of Proposition 6.2 and Proposition 6.4. □

7. Results obtained by interpolation

As was mentioned before, using our results concerning Besov-Lipschitz and Triebel-Lizorkin boundedness of FIOs, we can also extend the ranges of Triebel-Lizorkin boundedness a bit further. This is done by complex interpolation (see e.g. [6]) in the vertical direction between $F^s_{p,p}(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n)$ and $F^s_{p,2}(\mathbb{R}^n)$ (as in Figure 1).

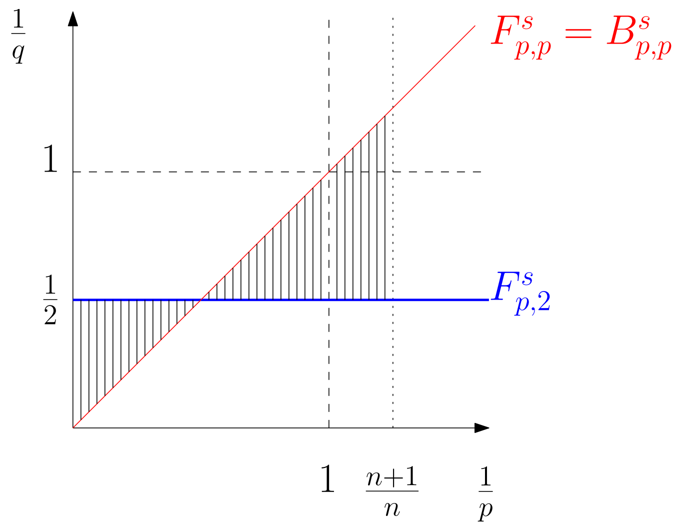

FIGURE 1. Global boundedness in Triebel-Lizorkin scale.

This yields the following theorem:

Theorem 7.1. *Let* $0 < p \leqslant \infty$, $m_c(p) := -(n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$, $a(x,\xi) \in S^{m_c}(\mathbb{R}^n)$, *and* $\varphi(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$, *be positively homogeneous of degree one in* ξ. *Then under these assumptions, the following results hold true*:

(i) *If* $a(x,\xi)$ *has compact support in* x *and* $\varphi(x,\xi)$ *is non-degenerate on the support of* $a(x,\xi)$, *then for any* $s \in \mathbb{R}$, $0 < p < \infty$, $\min(2,p) \leqslant q \leqslant \max(2,p)$, *the operator* T^φ_a *is bounded from* $F^s_{p,q}(\mathbb{R}^n)$ *to* $F^s_{p,q}(\mathbb{R}^n)$.

(ii) *If* $\varphi(x,\xi) \in \Phi^2$ *is* SND, *then for any* $s \in \mathbb{R}$, $\frac{n}{n+1} < p < \infty$, $\min(2,p) \leqslant q \leqslant \max(2,p)$, *the operator* T_a^φ *is bounded from* $F^s_{p,q}(\mathbb{R}^n)$ *to* $F^s_{p,q}(\mathbb{R}^n)$.

(iii) *In both cases* (i) *and* (ii) *the corresponding operator is bounded from* $F^s_{\infty,2}(\mathbb{R}^n)$ *to* $F^s_{\infty,2}(\mathbb{R}^n)$, *for* $s \in \mathbb{R}$.

(iv) *If* $\varphi(x,\xi) = |\xi| + x\cdot\xi$ *for* $s \in \mathbb{R}$ *and* $1 \leqslant q \leqslant \infty$ *one has that for* $a \in S^m(\mathbb{R}^n)$

$$\|T_a^{|\cdot|} f\|_{F^{s-m-\frac{n-1}{2}}_{1,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^s_{1,q}(\mathbb{R}^n}.$$

Statement (iii) is the consequence of the fact that for the aforementioned phases, the adjoint of the operator is bounded from $F^{-s}_{1,2}(\mathbb{R}^n)$ to $F^{-s}_{1,2}(\mathbb{R}^n)$.

The last claim follows from the work of J. Peral [15], which implies that for $a \in S^{-\frac{n-1}{2}}(\mathbb{R}^n)$ the operator $T_a^{|\cdot|} f$ has a factorisation $b(D)(f * d\sigma)$ where $d\sigma$ is the surface measure of the unit sphere and $b(\xi) \in S^0(\mathbb{R}^n)$. This in turn yields

$$\|T_a^{|\cdot|} f\|_{F^{s-m-\frac{n-1}{2}}_{1,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{F^s_{1,\infty}(\mathbb{R}^n)}.$$

and interpolation of this with $F^s_{1,2}(\mathbb{R}^n)$ yields the desired result.

Remark 7.2. *The* $F^s_{1,\infty}(\mathbb{R}^n)$ *result above concerning the phase functions of the form* $x\cdot\xi + |\xi|$ *could presumably be extended to a global result for phase functions of the form* $x\cdot\xi + \phi(\xi)$ (ϕ *positively homogeneous of degree* 1) *or a local regularity for operators with phases of the form* $\phi(x,\xi)$ (*positively homogeneous of degree* 1 *in* ξ *and non-degenerate*). *This is done by using a result of* T. Tao [22] *to decompose the corresponding* FIOs *into a composition of a pseudodifferential operator and and averaging operator. The details for this will appear elsewhere.*

8. Boundedness of FIOs on Triebel-Lizorkin spaces in dimension one

In this section we separate the results in dimension one that were missing in the previous section for Triebel-Lizorkin spaces. We will also see that one has much more flexibility in dimension one in proving the optimal results for all scales of the Triebel-Lizorkin spaces. To this end we have

Theorem 8.1. *Let* $p \in (0,\infty]$, $a(x,\xi) \in S^0(\mathbb{R})$ *and* $\varphi(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}\setminus\{0\})$, *be positively homogeneous of degree one in* ξ.
If $\varphi \in \Phi^2$ *and is* SND, *then* T_a^φ *is bounded from* $F^s_{p,q}(\mathbb{R})$ *to itself, for* $\frac{1}{2} < p \leqslant \infty$ *and* $0 < q \leqslant \infty$. *Once again, the assumption of the compact support of the amplitude in* x *and the non-degeneracy of the phase yields the result for the improved range* $p \in (0,\infty]$.

Proof. Let $a_+(\xi) \in C^\infty(\mathbb{R})$ such that $a_+(\xi) = 0$ when $\xi \leqslant 0$ and $a_+(\xi) = 1$ when $\xi \geqslant 1$ and let $a_-(\xi) := a_+(-\xi)$. Now write 1 as $a_+(\xi) + a_-(\xi) + r(\xi)$, where

$a_\pm \in S^0_{1,0}(\mathbb{R})$ and $r(\xi) = 1 - a_+(\xi) - a_-(\xi) \in C_c^\infty(\mathbb{R})$. Moreover using the (degree one) positive homogeneity of the phase function and the fact that we are in dimension one, we also have that $\varphi(x,\xi) = |\xi|\varphi(x, \operatorname{sgn}(\xi))$. This yields that

$$
\begin{aligned}
T_a^\varphi f(x) = &\int_{\mathbb{R}} a_+(x,\xi)\, e^{i\varphi(x,1)\xi}\, \widehat{f}(\xi)\, đ\xi + \int_{\mathbb{R}} a_-(x,\xi)\, e^{-i\varphi(x,-1)\xi}\, \widehat{f}(\xi)\, đ\xi \\
&+ \int_{\mathbb{R}} r_a(x,\xi)\, e^{i\varphi(x,\xi)}\, \widehat{f}(\xi)\, đ\xi,
\end{aligned}
$$

where $a_\pm(x,\xi) = a_\pm(\xi)\, a(x,\xi)$ and $r_a(x,\xi) = r(\xi)\, a(x,\xi)$. Therefore, using the invariance of $F^s_{p,q}(\mathbb{R})$ (with $0 < p < \infty$) under change of variables (observe that $|\varphi'(x,1)| \lesssim 1$ by the Φ^2 condition) and the boundedness of pseudodifferential operators on $F^s_{p,q}(\mathbb{R})$ together with Proposition 6.4 below, we obtain the $F^s_{p,q}(\mathbb{R})$ boundedness of the first two terms above. The boundedness of the third term is trivial as the amplitude of that operator belongs to $S^{-\infty}(\mathbb{R})$.

For $F^s_{\infty,q}(\mathbb{R})$, we use once again duality, which amounts to show that the adjoint operator

$$
\begin{aligned}
T_a^{\varphi *} f(x) = &\iint_{\mathbb{R}\times\mathbb{R}} \overline{a}_+(y,\xi)\, e^{i(x-\varphi(y,1))\xi} f(y)\, đ\xi\, \mathrm{d}y \\
&+ \iint_{\mathbb{R}\times\mathbb{R}} \overline{a}_-(y,\xi)\, e^{i(x+\varphi(y,-1))\xi} f(y)\, đ\xi\, \mathrm{d}y \\
&+ \iint_{\mathbb{R}\times\mathbb{R}} \overline{r}_a(y,\xi)\, e^{i(x\xi-\varphi(y,\xi))} f(y)\, đ\xi\, \mathrm{d}y.
\end{aligned}
$$

is bounded from $F^{-s}_{1,q'}(\mathbb{R})$ to itself where $\dfrac{1}{q'} + \dfrac{1}{q} = 1$. Therefore, once again the invariance of $h^1(\mathbb{R})$ under global diffeomorphisms with bounded Jacobians reduces the problem to show that a pseudodifferential operator of order zero the form

$$
\iint_{\mathbb{R}\times\mathbb{R}} \overline{b}(y,\xi)\, e^{i(x-y)\xi}\, f(y)\, đ\xi\, \mathrm{d}y
$$

is bounded on $F^{-s}_{1,q'}(\mathbb{R})$ which is well-known by e.g. [24]. The boundedness of the third term is trivial, once again due to the rapid decay of its amplitude. This concludes the proof of the theorem in the case of $0 < p \leqslant \infty$ in dimension one.

□

The following corollary yields the invariance of the Triebel-Lizorkin spaces $F^s_{\infty,q}(\mathbb{R})$ under change of variables, which is missing in the literature, see e.g. Theorem 2.5.

Corollary 8.2. *If η is a diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$ such that $|\eta'(x)| \sim 1$ for all $x \in \mathbb{R}$ then for $0 < q \leqslant \infty$ one has that*

$$
\|f \circ \eta\|_{F^s_{\infty,q}(\mathbb{R})} \lesssim \|f\|_{F^s_{\infty,q}(\mathbb{R})}.
$$

Proof. The result follows by observing that $f \circ \eta(x)$ can be expressed as an FIO with amplitude 1 and the phase function $\eta(x)\xi$, which verifies both the SND and the Φ^2 conditions and is therefore bounded on $F^s_{\infty,q}(\mathbb{R})$. □

9. Sharpness of the results

In this section we explain why the restriction imposed on p in Theorem 5.7 is necessary. To see this, if we let $\sigma \in S^{m_c(p)}(\mathbb{R}^n)$ be supported in a neighbourhood of the origin and take a function $f \in \mathscr{S}(\mathbb{R}^n)$ such that $\widehat{f}$ is equal to one on the support of $\sigma(\xi)$, and take $\psi_0 \in C_c^\infty(\mathbb{R}^n)$ such that it is equal to one on the support of $\widehat{f}$. Then we can take annuli-supported ψ_j's such that $\psi_j(D)T_\sigma^\phi f(x) = 0$ for $j \geqslant 1$ and

$$\psi_0(D)T_\sigma^\phi f(x) = \int_{\mathbb{R}^n} \sigma(\xi)\, e^{ix\cdot\xi + i\phi(\xi)}\, đ\xi.$$

Now assume that T_σ^ϕ is bounded on $B^s_{p,q}(\mathbb{R}^n)$ for all $p \in (0,\infty]$ then

$$\left\|T_\sigma^\phi f\right\|_{B^s_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

Moreover using the boundedness assumption above, Definition 2.2, the fact that $\psi_j(D)T_\sigma^\phi f(x) = 0$ for $j \geqslant 1$, and finally the frequency localisation of f yield that for all s, q and p one has

$$\begin{aligned}\left\|\int_{\mathbb{R}^n} \sigma(\xi)\, e^{ix\cdot\xi + i\phi(\xi)}\, đ\xi\right\|_{L^p(\mathbb{R}^n)} &= \|\psi_0(D)T_\sigma^\phi f\|_{L^p(\mathbb{R}^n)} = \|T_\sigma^\phi f\|_{B^s_{p,q}(\mathbb{R}^n)}\\ &\lesssim \|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \|\psi_0(D) f\|_{L^p(\mathbb{R}^n)}\\ &= \|f\|_{L^p(\mathbb{R}^n)} < \infty.\end{aligned}$$

But since

$$\int_{\mathbb{R}^n} \sigma(\xi)\, e^{ix\cdot\xi + i\phi(\xi)}\, đ\xi$$

is equal to the integral kernel $K(x)$ of the FIO T_σ^ϕ, then the decay provided by Lemma 2.14 which is actually sharp, won't yield $\|K\|_{L^p(\mathbb{R}^n)} \leqslant \|\langle\cdot\rangle^{-n-\varepsilon}\|_{L^p(\mathbb{R}^n)} < \infty$, for $p \in \left(0, \dfrac{n}{n+1}\right]$.

In dimension $n = 1$ we can explicitly see this by considering the FIO with amplitude identically equal to $1 \in S^0(\mathbb{R})$

$$Tf(x) := \int_{\mathbb{R}} \widehat{f}(\xi) e^{i|\xi| + ix\xi}\, đ\xi = \frac{f(x+1) + f(x-1)}{2} + i\frac{Hf(x+1) - Hf(x-1)}{2},$$

where the operator H is the Hilbert transform. If we take f to be the characteristic function of the interval $[-1, 1]$, one can calculate that

$$Hf(x) = \frac{1}{\pi}\log\left|\frac{x+1}{x-1}\right|.$$

This implies that the imaginary part of Tf is

$$\begin{aligned}\frac{Hf(x+1) - Hf(x-1)}{2} &= \frac{1}{2\pi}\left(\log\left|\frac{x+2}{x}\right| - \log\left|\frac{x}{x-2}\right|\right) = \frac{1}{2\pi}\log\left|1 - \frac{4}{x^2}\right|\\ &= -\frac{2}{\pi x^2} + O\left(x^{-4}\right)\end{aligned}$$

as $|x| \to \infty$. Note that $\log \left|1 - \dfrac{4}{x^2}\right| \in L^p_{\text{loc}}(\mathbb{R})$ for $0 < p < \infty$, but since the real part of Tf is compactly supported, the asymptotic expansion above yields that

$$Tf(x) = \left(\frac{\widehat{f}(0)}{\pi i}\right)\frac{1}{x^2} + O\left(x^{-4}\right)$$

as $|x| \to \infty$. From this, it follows that Tf can not be in $B^s_{p,q}(\mathbb{R})$ unless $p > \dfrac{1}{2}$.

The local result in Theorem 5.7 is sharp by the virtue of the sharpness of the classical Seeger-Sogge-Stein theorem [20].

10. Applications to Hyperbolic PDEs

In this section we outline some of the applications of the main results of this paper. This concerns local and global Besov-Lipschitz estimates for solutions to the Cauchy problems for strictly hyperbolic partial differential equations. First let us consider the basic example of the wave equation in $\mathbb{R}^{n+1}$

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = 0, \quad t \neq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = f_0(x), \\ \partial_t u(0,x) = f_1(x). \end{cases}$$

It is well-known that the solution to this Cauchy problem is given by

$$u(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|)}\left(\frac{\widehat{f_0}(\xi)}{2} + \frac{\widehat{f_1}(\xi)}{2i|\xi|}\right) đ\xi + \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|)}\left(\frac{\widehat{f_0}(\xi)}{2} - \frac{\widehat{f_1}(\xi)}{2i|\xi|}\right) đ\xi.$$

Now, using Theorem 5.7 it is not hard to verify that for some $\tau > 0$ and each $t \in [-\tau, \tau]$ and all $p \in \left(\dfrac{n}{n+1}, \infty\right]$, $0 < q \leqslant \infty$, $m \in \mathbb{R}$, $s \in \mathbb{R}$ and $m_c(p)$ as in (5) then

$$\sup_{t\in[-\tau,\tau]} \left\|(1-\Delta)^{\frac{m}{2}} u\right\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant C_\tau \left(\|f_0\|_{B^{s+m-m_c(p)}_{p,q}(\mathbb{R}^n)} + \|f_1\|_{B^{s+m-1-m_c(p)}_{p,q}(\mathbb{R}^n)}\right),$$

from which it follows that the solution of the wave equation verifies the following global (spatial) Besov space estimate

$$\sup_{t\in[-\tau,\tau]} \|u\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant C_\tau \left(\|f_0\|_{B^{s+(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|}_{p,q}(\mathbb{R}^n)} + \|f_1\|_{B^{s+(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|-1}_{p,q}(\mathbb{R}^n)}\right). \tag{58}$$

In particular, for $p = q$ and $s \in \mathbb{R}\setminus\mathbb{Z}$ (i.e. non-integer), (58) is the global extension of the Sobolev and Lipschitz space estimates in Theorem 4.1 of [20], for the case of wave equation. Moreover (58) goes beyond that result since it also provides estimates for the solution in quasi-Banach spaces.

Similarly, using Theorem 7.1 we have for any $s \in \mathbb{R}$, $\dfrac{n}{n+1} < p < \infty$, $\min(2,p) \leqslant q \leqslant \max(2,p)$ that

$$\sup_{t\in[-\tau,\tau]} \|u\|_{F^s_{p,q}(\mathbb{R}^n)} \leqslant C_\tau \left(\|f_0\|_{F^{s+(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|}_{p,q}(\mathbb{R}^n)} + \|f_1\|_{F^{s+(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|-1}_{p,q}(\mathbb{R}^n)}\right). \tag{59}$$

Moreover if $p = 1$ then the estimate above can actually be extended to the whole range $1 \leqslant q \leqslant \infty$, and if $p = \infty$ and $q = 2$ then the estimate still holds true, in particular one has

$$\sup_{t\in[-\tau,\tau]} \|u\|_{\mathrm{bmo}(\mathbb{R}^n)} \leqslant C_\tau \left(\|f_0\|_{F^{\frac{n-1}{2}}_{\infty,2}(\mathbb{R}^n)} + \|f_1\|_{F^{\frac{n-3}{2}}_{\infty,2}(\mathbb{R}^n)}\right),$$

which yields that in 3 spatial dimensions,

$$\sup_{t\in[-\tau,\tau]} \|u\|_{\mathrm{bmo}(\mathbb{R}^3)} \leqslant C_\tau \left(\|f_0\|_{F^1_{\infty,2}(\mathbb{R}^3)} + \|f_1\|_{\mathrm{bmo}(\mathbb{R}^3)}\right).$$

Concerning the local Besov space estimates, one can improve on the range of the estimates in p. In this connection let us consider the Cauchy problem for a strictly hyperbolic partial differential equation

$$\begin{cases} D_t^N u + \sum_{j=1}^N P_j(x,t,D_x) D_t^{N-j} u = 0, & t \neq 0 \\ \partial_t^j u\big|_{t=0} = f_j(x), & 0 \leqslant j \leqslant N-1, \end{cases} \tag{60}$$

for $N \in \mathbb{N}$, it is well-known (see e.g. [21]) that this problem can be solved locally in time and modulo smoothing operators by

$$u(x,t) = \sum_{j=0}^{N-1} \sum_{k=1}^{N} \int_{\mathbb{R}^n} e^{i\varphi_k(x,\xi,t)} a_{jk}(x,\xi,t)\, \widehat{f}_j(\xi)\, đ\xi, \tag{61}$$

where $a_{jk}(x,\xi,t)$ are suitably chosen amplitudes depending smoothly on t and belonging to $S^{-j}_{1,0}(\mathbb{R}^n)$, and the phases $\varphi_k(x,\xi,t)$ also depend smoothly on t, are strongly non-degenerate and belong to the class Φ^2. This yields the following:

Theorem 10.1. *Let $u(x,t)$ be the solution of the hyperbolic Cauchy problem* (60) *with initial data f_j. Then for all $p,q \in (0,\infty]$ and $s \in \mathbb{R}$ and any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, the solution $u(\cdot,t)$ satisfies the local Besov-Lipschitz space estimate*

$$\sup_{t\in[-\tau,\tau]} \|\chi\, u\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant C_\tau \sum_{j=0}^{m-1} \|f_j\|_{B^{s+(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|-j}_{p,q}(\mathbb{R}^n)}. \tag{62}$$

Similarly for any $s \in \mathbb{R}$, $0 < p < \infty$, $\min(2,p) \leqslant q \leqslant \max(2,p)$, one has the local Triebel-Lizorkin estimate

$$\sup_{t\in[-\tau,\tau]} \|\chi\, u\|_{F^s_{p,q}(\mathbb{R}^n)} \leqslant C_\tau \sum_{j=0}^{m-1} \|f_j\|_{F^{s+(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|-j}_{p,q}(\mathbb{R}^n)}, \tag{63}$$

Which also holds when $p = \infty$ and $q = 2$. Moreover if $m < -(n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$ then for all $s \in \mathbb{R}$ and $p,q \in (0,\infty]$ one has

$$\sup_{t\in[-\tau,\tau]} \|\chi\, u\|_{F^s_{p,q}(\mathbb{R}^n)} \leqslant C_\tau \sum_{j=0}^{m-1} \|f_j\|_{F^{s-m-j}_{p,q}(\mathbb{R}^n)}.$$

Furthermore, all the estimates above can be globalised (*i.e. we can remove the cut-off function χ in all of them*) *for $p \in \left(\frac{n}{n+1}, \infty\right]$, $q \in (0,\infty]$ and $s \in \mathbb{R}$.*

Proof. This follows at once from the Fourier integral operator representation (61) and theorems 5.7, 7.1 and 6.1. □

Estimate (62) is an extension of (2) which was proven in [20], to the case of $s \in \mathbb{R}$, $p \neq q$ and also the quasi-Banach setting.

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