A note on commutator estimates for interpolation methods

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Abstract

The aim of this note is to show how all the commutator estimates of two recent papers, by M. Cwikel, N. Kalton, M. Milman and R. Rochberg and by N. Krugljak and M. Milman, can be considered as special cases of the method of couples of interpolators introduced by M.J. Carro, J. Cerdà and J. Soria, and also to show how the distance between orbits and the “Benson norm” considered by Krugljak and Milman can be extended, respectively, as a distance between the interpolators that appear in the general construction and as a constant that is finite when the interpolators satisfy the necessary cancellation property.

1 Introduction

Interpolation theory of operators plays a central role in certain commutator estimates of Analysis. In [19], Rochberg and Weiss developed the study of the commutators of bounded linear operators and certain operators, generally unbounded and nonlinear, associated with the complex interpolation method, with very interesting application to estimates for commutators of singular integrals with pointwise multipliers.

A similar construction was done in [13] by Jawerth, Rochberg and Weiss for the real method, and further results and applications to classical analysis have been obtained in [11], [10], [14], [15], [4], [7], [8] and [9], among others.

Initially, in [19] and [13], the constructions were essentially developed separately for real and complex methods, but soon (see [11]) it was asked whether both approaches could be unified by a single general method.

In [5] and [6], a way of doing a unified treatment was shown using the construction of interpolation spaces presented by Williams in [21]. This approach allowed to prove a very general commutator theorem based on the isolation of the needed cancellation property, in the line of [17] and [4], and includes the usual interpolation methods.

Very recently, new theories for commutator estimates have been presented in [12] and [16].

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The unifying approach of [12] was suggested by the work of Peetre in [18] and its main feature is the systematic use of derivatives of analytic functions on the annulus, in the spirit of the earlier work on commutators, combined with the cancellations provided by the derivatives.

The paper [16] is devoted to a version of Shneiberg’s theorem about invertible operators and to extend the commutator theorem for orbital methods, with the use of a distance between orbit spaces generated by a single element.

The goal of this note is twofold. First, to show how all the commutator estimates of [12] and [16] can be considered as special cases of the method introduced in [5], including an answer to a question posed in [12] concerning the commutators of translation mappings. On the other hand, to show how the distance between orbits and the “Benson norm” of [16] can be extended, respectively, as a distance between the interpolators that appear in the general construction and as a constant that is finite precisely when the interpolators satisfy the necessary cancellation property.

Of course, the more specific the interpolation methods are, the more precise can estimates, distances and constants be made, and part of the richness of the theory depends on the concrete interpolation methods. But it is our feeling that our construction can be useful to unify and simplify the possible results concerning commutator estimates.

We shall follow the usual notations of interpolation theory. For instance, $X = (X_0, X_1)$ will be a Banach couple, and by $T \in L(X; Y)$ or $T : X \to Y$ we represent a bounded linear mapping between two Banach couples in the sense that $T : \Sigma X \to \Sigma Y$ is a linear mapping ($\Sigma X$ represents the sum space $X_0 + X_1$) such that $T(X_j) \subset Y_j$ ($j = 0, 1$) and $\|T\| := \max(\|T\|_{X_0, Y_0}, \|T\|_{X_1, Y_1}) < \infty$.

For general facts on interpolation theory, we refer to [1] and [2].

2 Interpolators

An interpolation method in the sense of Definition 2.1 of [5] is given by a functor $H$ from a class of Banach couples $\mathcal{X}$ to Banach spaces (usually “functional spaces”, spaces of vector valued functions or sequences) and an interpolator $\Phi$ over $H$.

This means that, for each Banach couple $\mathcal{X}$, we have a bounded linear mapping $\Phi_\mathcal{X} : H(\mathcal{X}) \to \Sigma \mathcal{X}$ such that, if $T : \mathcal{X} \to \mathcal{Y}$, then $T \circ \Phi_\mathcal{X} = \Phi_\mathcal{Y} \circ H(T)$.

Of course, $H(T) : H(\mathcal{X}) \to H(\mathcal{Y})$ is the bounded linear mapping associated to $T$. We will assume that $\|H(T)\| \leq \|T\|$, since this will happen with all the examples.

Under the above conditions, we denote

$X_\Phi = \text{Im} \Phi_\mathcal{X} \simeq H(\mathcal{X}) / \text{Ker} \Phi_\mathcal{X}$,

dowered with its natural norm, $\|\Phi_\mathcal{X}(h)\|_\Phi = d(h, \text{Ker} \Phi_\mathcal{X})$. Then, $\mathcal{X} \mapsto X_\Phi$ is an interpolation method and, for every $T : \mathcal{X} \to \mathcal{Y}$, $\|H(T)\| \leq \|T\|_{X_\Phi, Y_\Phi}$.

The model example is the first method of Calderón for complex Banach spaces. In this case, for a given $\theta \in (0, 1)$, $H(\mathcal{X}) = \mathcal{F}(\mathcal{X})$ is a Banach space of analytic $\Sigma \mathcal{X}$-valued functions on the strip $0 < \Re z < 1$, $H(T)F = T \circ F = T(F)$ and $\Phi(F) = \delta_\theta(F) = F(\theta)$.
Another interpolator over the same functional spaces is the Lions-Schechter method of derivatives, given by \( \delta'_0(F) = F'(\theta) \).

We refer to [5] for a description of couples of interpolators associated to the real \( K \)– methods, and to the equivalent \( J \)– interpolation methods. These methods may also be introduced using the pseudolattices construction of [12] and many of them as orbits of a single element, as in [16].

For the sake of simplicity, let us summarize the construction of [12] under some weak restrictions that are satisfied by the usual examples.

If \( X \) is a complex Banach space and \( E \) a nontrivial pseudolattice in the sense of [12], \( E(X) \) is a Banach space of sequences \( \{x_k\}_{k \in \mathbb{Z}} \subset X \) such that

1) If \( Y \) is a closed subspace of \( X \), \( E(Y) \) is a closed subspace of \( E(X) \),

2) \( \|x_m\|_X \leq \|\{x_k\}_{k \in \mathbb{Z}}\|_{E(X)} \),

3) \( \|\{T x_k\}_{k \in \mathbb{Z}}\|_{E(Y)} \leq \|T\|\|\{x_k\}_{k \in \mathbb{Z}}\|_{E(X)} \) for any bounded linear mapping \( T : X \to Y \), and

4) \( \delta_0 = \{\delta^m_k\}_{k \in \mathbb{Z}} \in E(\mathbb{C}) \).

We are going to suppress the subscript \( k \in \mathbb{Z} \) when dealing with sequences, and we write \( \{x_k\} \) instead of \( \{x_k\}_{k \in \mathbb{Z}} \).

We say that \( E \) is regular if the shift operator \( S\{x_k\} := \{x_{k-1}\} \) is an isometry from \( E(X) \) onto itself, for every \( X \).

As an example, consider the pseudolattice \( F\mathcal{C} \). For every complex Banach space \( X \), \( F\mathcal{C}(X) \) denotes the Fourier transform of \( \mathcal{C}(\mathbb{T} \cap X) \), the \( X \)-valued continuous periodic functions, endowed with the “sup norm” of these functions.

When \( E \) is a sequence lattice, we denote \( E(X) \) the linear space of all sequences \( \{x_k\} \subset X \) such that \( \{\|x_k\|_X\} \in E \), and \( \|\{x_k\}\|_{E(X)} := \|\{\|x_k\|_X\}\|_E \). If \( \|\{\delta^m_k\}\|_{E(\mathbb{C})} \leq 1 \) for every \( m \in \mathbb{Z} \), then \( X \to E(X) \) is a nontrivial pseudolattice.

Let \( \bar{E} \) be a fixed couple of nontrivial pseudolattices. For each Banach couple \( \bar{X}, \mathcal{J}(\bar{X}) \) denotes the Banach space all \( X_0 \cap X_1 \)-valued sequences \( \{x_k\} \) such that \( \{x_k\} \in E_0(X_0) \) and \( \{e^k x_k\} \in E_1(X_1) \), endowed with the norm

\[
\|\{x_k\}\|_{\mathcal{J}(\bar{X})} := \max(\|\{x_k\}\|_{E_0(X_0)}, \|\{e^k x_k\}\|_{E_1(X_1)}).
\]

If \( \{x_k\} \in \mathcal{J}(\bar{X}) \), condition \( \|\{x_k\}\|_{\mathcal{J}(\bar{X})} < \infty \) ensures the absolute convergence of \( \sum_{k<0} z^k x_k \) in \( X_0 \) for \( |z| > 1 \), and of \( \sum_{k\geq 0} z^k x_k \) in \( X_1 \) for \( |z| < \epsilon \). Hence, \( \sum_k z^k x_k \) defines an analytic \( \Sigma \bar{X} \)-valued function on the annulus \( \mathbb{A} := \{z \in \mathbb{C}; 1 < |z| < \epsilon\} \) and we will use the notation \( x(z) := \sum_{k\geq 0} z^k x_k \), so that \( x'(z) = \sum_k k x_k z^{k-1} \).

If \( T : \bar{X} \to \bar{Y} \), then \( \mathcal{J}(T) : \mathcal{J}(\bar{X}) \to \mathcal{J}(\bar{Y}) \) is the linear mapping such that \( \mathcal{J}(T)(\{x_k\}) := T\{x_k\} \in \mathcal{J}(\bar{Y}) \). We remark that \( \|\mathcal{J}(T)\| \leq \|T\| \).

Let \( s \in \mathbb{A} \). The evaluation, \( \delta_s(\{x_k\}) = x(s) \), is an interpolator over \( \mathcal{J} \) and, as in [12], we denote \( \bar{X}_{\delta_s} \) the interpolated space \( \bar{X}_{\delta_s} \). Of course, this defines an interpolation method and, under our assumptions, it is proved in Theorem 2.14 of [12] that \( \|T\|_{\bar{X}_{\delta_s},\bar{Y}_{\delta_s}} \leq e\|T\|_{X_0,Y_0}\|T\|_{X_1,Y_1}^{\frac{1}{\vartheta}} \), with \( \vartheta = \log |s| \).
Up to equivalence of norms, for suitable elections of $E$ and $s$, this interpolation method coincides with the usual real and complex methods: $\hat{X}_{(\ell^p,\ell^q),e^\theta} = \hat{X}_{p,\theta}$, the Lions-Peetre real method, and $\hat{X}_{(F\ell,C\ell),e^\theta} = \hat{X}_{[\theta]}$, the first Calderón method. Also, the Peetre $\pm$ method and its Gustavson-Peetre variant are obtained in a similar way.

For an orbital method generated by a single element, a couple $\hat{A}$ of Banach spaces and an element $a \in \Sigma \hat{A}$ must be fixed. Then $\mathcal{H}(\hat{X}) = \mathcal{L}(\hat{A}; \hat{X})$ (or a subspace of $\mathcal{L}(\hat{A}; \hat{X})$ with the left ideal property) are the functional spaces, $\mathcal{H}(T)(R) := T \circ R$ for every $R \in \mathcal{L}(\hat{A}; \hat{X})$, and the interpolator is the evaluation $\delta_a$, so, $\delta_a(R) = Ra \in \Sigma \hat{X}$. Then, the interpolated space is

$$\text{Orb}_a(\hat{X}) := \delta_a(\mathcal{H}(\hat{X})) \simeq \mathcal{H}(\hat{X})/\text{Ker}_a(\hat{X}),$$

with $\text{Ker}_a(\hat{X}) = \{ R \in \mathcal{H}(\hat{X}); Ra = 0 \}$.

Again, classical interpolation methods are obtained as special cases. For instance, if $a_{\theta}(t) := \theta(1 - \theta)t^\theta$ and $\hat{A} = (L^1((0, \infty), dt), L^1((0, \infty), dt/t))$, then $\text{Orb}_{a_{\theta}}(\hat{X}) = \hat{X}_{\theta,\infty}$.

If $1 \leq q < \infty$, for $\hat{A} = (\ell^1, \ell^1(2^{-n}))$ and $a_{\theta} = \{2^{-n}\}_{n \in \mathbb{N}}$, $\text{Orb}_{a_{\theta}}(\hat{X}) = \hat{X}_{\theta,2}$, but now $\mathcal{H}(\hat{X})$ is the left ideal of all $R \in \mathcal{L}(\hat{A}; \hat{X})$ such that

$$||R|| := \max \left( \left( \sum \|R(e_n)\|_{X_0}^q \right)^{1/q}, \left( \sum \|R(2^n e_n)\|_{X_1}^q \right)^{1/q} \right) < \infty,$$

where $e_n := \{\delta_n^k\}$.

Also, if $FL^\theta$ denotes the space of all complex sequences $\{a_k\}$ such that $a_k e^{-k\theta}$ are the Fourier coefficients of a function $f \in L^1(\mathbb{T})$, normed by $||f||_1$, the second complex interpolation method of Calderón is $\hat{X}^{[\theta]} = \text{Orb}_{a_{\theta}}(\hat{X})$ if $\hat{A} = (FL^0, FL^1)$, $a_{\theta} = \{e^{\theta n}\}$ and $\mathcal{H}(\hat{X}) = \mathcal{L}(\hat{A}, \hat{X})$.

Finally, the first complex interpolation method $\hat{X}^{[\theta]}$ is obtained in a similar way, $\mathcal{H}(\hat{X})$ beeing the closure in $\mathcal{L}(\hat{A}, \hat{X})$ of all operators $R$ such that $R(\Sigma \hat{A}) \subset X_0 \cap X_1$ (cf. Theorem 15 in [16]).

### 3 Distance function between interpolators

If $\Phi$ and $\Psi$ are two interpolators on the same functional spaces $\mathcal{H}(\hat{X})$, *mutatis mutandis*, the definition given in [16] for orbital methods can be extended to a distance function between $\Phi$ and $\Psi$,

$$\varrho_\mathcal{H}(\Phi, \Psi) := \sup_{||f||_{\mathcal{H}(\hat{X})} \leq 1} \left( ||\Phi(\hat{X})(f)||_\Phi - ||\Psi(\hat{X})(f)||_\Psi \right),$$

equation i.e.,

$$\varrho_\mathcal{H}(\Phi, \Psi) = \sup_{||f||_{\mathcal{H}(\hat{X})} \leq 1} \left( \left| d(f, \text{Ker}\Phi) - d(f, \text{Ker}\Psi) \right| \right),$$

so that, if $B(\Phi_X)$ and $B(\Psi_X)$ are the closed unit balls of $\text{Ker}\Phi_X$ and $\text{Ker}\Psi_X$, then it is easily seen that

$$\frac{1}{2} \varrho^h(B(\Phi_X), B(\Psi_X)) \leq \varrho_\mathcal{H}(\Phi, \Psi) \leq 2 \varrho^h(B(\Phi_X), B(\Psi_X)),$$

where $\varrho^h$ denotes the Hausdorff distance.
where \( \varrho^h \) denotes the Hausdorff distance between closed sets of \( H(\bar{X}) \),
\[
\varrho^h(B(\Phi_X), B(\Psi_X)) = \max \left( \sup_{u \in B(\Phi_X)} d(u, B(\Psi_X)), \sup_{v \in B(\Psi_X)} d(v, B(\Phi_X)) \right).
\]

In the case of orbital methods generated by one element, the notations
\[
\varrho_{\bar{X}}(a, b) := \varrho_{\bar{X}}(\delta a, \delta b) = \sup_{\| R \|_{\mathcal{L}(\bar{A} ; \bar{X})} \leq 1} \| Ra \|_{\text{Orb}_a(\bar{X})} - \| Rb \|_{\text{Orb}_b(\bar{X})},
\]
\[
\varrho(a, b) := \sup_{\bar{X}} \varrho_{\bar{X}}(a, b)
\]
are used in [16], where some estimates are found for the example of the Lions-Peetre real method.

It is clear from that paper that this distance function can be extended to a more general setting. For the first complex method, if \( \delta \theta \) is the evaluation interpolator on the Calderón spaces \( \mathcal{F}(\bar{X}) \), the authors observe that [20] and [3] contain an estimate that can be written as
\[
\varrho_{\bar{X}}(\delta \theta_0, \delta \theta_1) \leq \left| F(\theta_0) - F(\theta_1) \right| \frac{1 - F(\theta_0)F(\theta_1)}{1 - F(\theta_1)},
\]
where \( F \) is a conformal mapping from the strip \( S \) onto the unit disc. Moreover, at the end of the paper, they also consider the distance function for the method of pseudolattices of [12].

### 4 Commutator estimates

Let \( \Phi \) be a fixed interpolator over \( H \), and \( x \in \bar{X}_\Phi \mapsto h_x \in H(\bar{X}) \) an almost optimal selection, with constant \( c_\Omega > 1 \), in the sense that
\[
\Phi_{\bar{X}}(h_x) = x, \quad \| h_x \|_{H(\bar{X})} \leq c_\Omega \| x \|_{\Phi}.
\]

Then, the “derivation” operators \( \Omega_{\bar{X}} \) associated to a second interpolator \( \Psi \) over \( H \) are defined in [5] as
\[
\Omega_{\bar{X}}(x) := \Psi_{\bar{X}}(h_x).
\]

They can be nonlinear, but \( \Omega_{\bar{X}} : \bar{X}_{\Phi} \rightarrow \bar{Y}_{\Psi} \) and \( \| \Omega_{\bar{X}}(x) \|_{\Psi} \leq c_\Omega \| x \|_{\Phi} \).

For instance, if \( \delta \theta \) is the interpolator on \( \mathcal{F}(\bar{X}) \) of our the model example, and
\[
x \in \bar{X}_{\delta \theta} = \bar{X}_{[\theta]} \mapsto h_x \in \mathcal{F}(\bar{X})
\]

an almost optimal selection, the operator \( \Omega_{\bar{X}} \) associated to the Lions-Schechter interpolator \( \delta'_\theta \) is \( \Omega_{\bar{X}}(x) = h'_x(\theta) \), a true derivation.

If \( T : \bar{X} \rightarrow \bar{Y} \), the commutator with \( \Omega \) is the operator \([T, \Omega] : \bar{X}_\Phi \rightarrow \bar{Y}_\Psi \) such that
\[
[T, \Omega](x) := T \Omega_{\bar{X}}(x) - \Omega_{\bar{Y}}(Tx) = T \Psi_{\bar{X}}(h_x) - \Psi_{\bar{Y}}(h_{Tx}) = \Psi_{\bar{Y}}(H(T)h_x - h_{Tx}).
\]

The obvious estimate \( \| [T, \Omega]x \|_{\Psi} \leq \| H(T)h_x - h_{Tx} \|_{H(\bar{Y})} \leq 2c_\Omega \| T \| \| x \|_{\Phi} \) (we are assuming that \( \| H(T) \| \leq \| T \| \)) may be precised by using the distance function \( \varrho_{\bar{Y}} \).
**Theorem 1** \( \| [T, \Omega] \|_{X_{s}, Y_{s}} \leq 2c_{\Omega} \| T \| \varrho_{Y}(\Phi, \Psi). \)

**Proof.** If \( h := H(T)h_{x} - h_{Tx} \), then \( \Psi_{Y}(h) = T\Phi_{X}(h_{x}) - \Phi_{Y}(h_{Tx}) = Tx - Tx = 0 \), so that \( h \in \text{Ker} \Phi_{Y} \) and
\[
\| \Psi_{Y}(h) \|_{\Psi} = d(h, \text{Ker} \Psi_{Y}) \leq \| h \|_{H(\Psi)} \varrho_{Y}(\Phi, \Psi),
\]
since, from the definition of \( \varrho_{Y} \), \( \varrho_{Y}(\Phi, \Psi) \geq d(g, \text{Ker} \Psi_{Y}) \) if \( \| g \|_{H(\Psi)} \leq 1. \)

Finally, as before, \( \| h \|_{H(\Psi)} \leq 2c_{\Omega} \| T \| \| x \|_{\Psi}. \)

Let us now go to special cases.

For a couple \( E \) of nontrivial pseudolattices, and \( s \neq s' \) in the annulus \( A \), we consider two evaluation interpolators \( \delta_s \) and \( \delta_{s'} \) over the functional spaces \( \mathcal{J}(X) \) and an almost optimal selection \( x \in X_{E,s} \mapsto \{x_{k}^{\Omega}\} \in \mathcal{J}(X) \) for the first one, so that
\[
x^{\Omega}(s) = \sum_{k} s^{k} x_{k}^{\Omega} = x, \quad \| \{x_{k}^{\Omega}\} \|_{\mathcal{J}(X)} \leq c_{\Omega} \| x \|_{\text{Ker} \Phi_{s}},
\]
The translation operators \( \mathcal{R}_{X} \) are the “derivation” operators corresponding to \( (\delta_{s}, \delta_{s'}) \),
\[
\mathcal{R}_{X}(x) := \delta_{s'}((\{x_{k}^{\Omega}\})_{x}) = x^{\Omega}(s').
\]
Hence, Theorem 1 applies and we obtain the commutator estimates of \( [12] \) for these operators in our setting in terms of the distance function \( \varrho(\delta_{s}, \delta_{s'}) \), without any regularity condition on \( E \).

**Corollary 1** \( \mathcal{R}_{X} : X_{E,s} \to X_{E,s'} \) and \( [T, \mathcal{R}] : X_{E,s} \to Y_{E,s'} \) if \( T \in \mathcal{L}(X; Y) \), with
\[
\| [T, \mathcal{R}] \| \mathcal{J}(X) \leq 2c_{\Omega} \varrho_{Y}(\delta_{s}, \delta_{s'}) \| T \|.
\]
Similarly, in the case of orbital methods generated by a single element, Proposition 4 in \( [16] \) is an special case of our Theorem 1. If \( a, b \in \Sigma \tilde{A} \), and \( x \in \text{Orb}_{a}(\tilde{X}) \to R_{x} \in \mathcal{L}(\tilde{A}; \tilde{X}) \) is an almost optimal selection with constant \( c_{\Omega} > 1 \), \( \Omega_{X}(x) = R_{x}(b) \).

**Corollary 2** \( \Omega_{X} : \text{Orb}_{a}(\tilde{X}) \to \text{Orb}_{b}(\tilde{X}) \) and, for any \( T \in \mathcal{L}(\tilde{X}; \tilde{Y}) \), \([T, \Omega] : \text{Orb}_{a}(\tilde{X}) \to \text{Orb}_{b}(\tilde{Y}) \) with
\[
\| [T, \Omega] \| \text{Orb}_{a}(\tilde{X}), \text{Orb}_{b}(\tilde{Y}) \leq 2c_{\Omega} \varrho_{Y}(a, b) \| T \|.
\]

**5 Cancellation**

As in \( [5] \), we say that the couple \( (\Phi, \Psi) \) of interpolators over \( H \) is **almost compatible** when, if \( \Phi_{X}(f) = 0 \), there exists \( h \in H(\tilde{X}) \) such that
\[
\Psi_{X}(f) = \Phi_{X}(h), \quad \| h \|_{H(\tilde{X})} \leq C_{\tilde{X}} \| f \|_{H(\tilde{X})}
\]
for some constant \( C_{\tilde{X}} > 0 \). This is a cancellation condition that means that the restriction of \( \Psi_{X} \) to \( \text{Ker} \Phi_{X} \) is a bounded operator
\[
\Psi_{X} : \text{Ker} \Phi_{X} \to \tilde{X}_{\Phi}.
\]
Thus, \( \| \Psi_{X} \|_{\text{Ker} \Phi_{X}, \tilde{X}_{\Phi}} < \infty \).

For these interpolators the commutator theorem of \( [5] \), \([T, \Omega] : \tilde{X}_{\Phi} \to \tilde{Y}_{\Phi} \), follows very easily:
If \((\Phi, \Psi)\) is almost compatible, then
\[
\| [T, \Omega] \|_{\mathcal{X}_{\Phi}, \mathcal{Y}_{\Phi}} \leq 2c_\Omega \| T \| \mathcal{X}_{\Phi} \| \Psi_\Phi \|_{\mathcal{Y}_{\Phi}} < \infty.
\]

**Proof.** In the proof of Theorem 1, \(\| \Psi_F (h) \|_{H(Y)} \leq \| h \|_{H(X)} \| \Psi_F \| \mathcal{Y}_{\Phi} \|_{\mathcal{X}_{\Phi}}\), since \(h \in \text{Ker} \Phi \).

In the special case of an orbital method generated by a single element \(a \in \Sigma \bar{A}\), in [16], the authors associate to \(a\) the so-called Benson space \(B(a)\). The Benson norm of \(b \in \Sigma \bar{A}\) is
\[
\| b \|_{B(a)} := \| \delta_b \|_{\text{Ker} \delta_a, \text{Orb} a},
\]
and
\[B(a) := \{ b \in \Sigma \bar{A}; \| b \|_{B(a)} < \infty \}.
\]
Hence, in our terminology, \(b \in B(a)\) means that \((\delta_a, \delta_b)\) is almost compatible.

Thus, Theorem 3 in [16],
\[
\| [T, \Omega] \|_{\text{Orb} a, \text{Orb} b} \leq 2c_\Omega \| T \| \| b \|_{B(a)},
\]
appears as a special instance of Theorem 2.

For the pseudolattices construction, as observed in [12], the regularity condition (which is sufficient for \(E\) to “admit differentiation uniformly”) implies almost compatibility of \((\delta_s, \delta_s')\), and in [16] the distance between \(\delta_s\) and \(\delta_s'\) has been estimated. For the sake of an easy reading, we reframe carefully and include the proof of these results in our setting and with our notations.

**Proposition 1.** Let us fix a regular couple, \(E\), of nontrivial pseudolattices, and \(1 < |s| < e\). Then, on every Banach couple \(\bar{X}\), the pair \((\delta_s, \delta_s')\) of interpolators over \(J\) is almost compatible, and
\[
\| \delta_{s}' \|_{\text{Ker} \delta_s, \mathcal{X}_{E, s}} \leq c(s), \quad c(s) = \max \left\{ \frac{1}{|s| - 1}, \frac{1}{e - |s|} \right\}.
\]

**Proof.** Let \(x(z) = \sum_k z^k x_k\), where \(\{x_k\} \in \mathcal{J}(\bar{X})\), be such that \(x(0) = 0\). Then
\[
y(z) := \frac{x(z)}{z - s} = \sum_{k = -\infty}^{+\infty} z^k y_k, \quad y_k = \sum_{n \geq 0} s^n x_{k+n+1} = -\sum_{n < 0} s^n x_{k+n+1}
\]
since, using the absolute convergence of the series, if \(|z| > |s|\),
\[
\frac{x(z)}{z - s} = \frac{1}{z} \sum_{n \geq 0} \left( \frac{s}{z} \right)^n \sum_{k = -\infty}^{+\infty} z^k x_k = \sum_{n \geq 0} \sum_{k = -\infty}^{+\infty} z^{k-n+1} s^n x_k = \sum_{k = -\infty}^{+\infty} z^k \sum_{n \geq 0} s^n x_{k+n+1}
\]
and, if \(|z| < |s|\),
\[
\frac{x(z)}{z - s} = -\frac{1}{s} \sum_{n \geq 0} \left( \frac{z}{s} \right)^n \sum_{k = -\infty}^{+\infty} z^k x_k = -\sum_{k = -\infty}^{+\infty} z^k \sum_{n < 0} s^n x_{k+n+1}.
\]
But \(S^{n+1}\) are isometries and, from \(y_k = \sum_{n \in \mathbb{Z}} (S^{n+1} \{s^n x_m\})_{m \in \mathbb{Z}}\), we obtain
\[
\| \{y_k\} \|_{E_0(X_0)} \leq \sum_{n < 0} |s|^n \| \{x_k\} \|_{E_0(X_0)} = \frac{1}{|s| - 1} \| \{x_k\} \|_{E_0(X_0)}.
\]
In the same way, since \( e^k y_k = (1/e) \sum_{n \geq 0} (s/e)^n e^{k+n+1} x_{k+n+1} \),

\[
||\{e^k y_k\}||_{E_1(X_1)} \leq \frac{1}{e} \left( 1 - \frac{|s|}{e} \right) ||\{e^k x_k\}||_{E_1(X_1)}
\]

Thus, \( ||y_k||_{J(\bar{X})} = \max(\{y_k\}|_{E_0(X_0)}, ||e^k y_k||_{E_1(X_1)}) \leq c(s)||\{x_k\}||_{J(\bar{X})} \). Obviously, \( x'(s) = y(s) \).

**Prop 2** If \( \tilde{E} \) is a regular couple of nontrivial pseudolattices and \( 1 < |s|, |s'| < e \), then

\[
\rho_X(\delta_s, \delta_{s'}) \leq c(s, s'), \quad c(s, s') = 2 \max\{c(s), c(s')\}
\]

**Proof.** Recall that \( \rho_X(\delta_s, \delta_{s'}) = \sup\{||x_k||_{J} \leq 1 ||x(s)|| \delta_s - ||x(s')|| \delta_{s'} \} \).

If \( u = x(s) \) with \( ||\{x_k\}||_{J} \leq 1 \), we may choose \( \tilde{x}_k \in J(\bar{X}) \) so that

\[
\tilde{x}(s) = u = x(s), \quad ||\tilde{x}_k||_{J} \simeq ||u|| \delta_s (||\{x_k\}||_{J} \leq (1 + \varepsilon)||x(s)|| \delta_s).
\]

Since \( x(s) - \tilde{x}(s) = 0 \), it follows as in Proposition 1 that \( x(z) - \tilde{x}(z) = (z - s)\tilde{g}(z) \) with \( \{y_k\} \in J(\bar{X}) \) and \( ||\{y_k\}||_{J} \leq c(s)||\{x_k\} - \{\tilde{x}_k\}||_{J} \leq 2c(s) \).

Thus,

\[
||x(s') - \tilde{x}(s')||_{\delta_{s'}} \leq |s - s'||y(s')||_{\delta_{s'}} \leq |s' - s||y_k||_{J} \leq 2c(s)|s' - s|
\]

and then

\[
||x(s')||_{\delta_{s'}} \leq ||\tilde{x}(s')||_{\delta_{s'}} + 2c(s)|s' - s| \leq ||\{\tilde{x}_k\}||_{J} + 2c(s)|s - s'|
\]

so that, for every \( \varepsilon > 0 \), we obtain

\[
||x(s')||_{\delta_s} \leq (1 + \varepsilon)||x(s)||_{\delta_s} + 2c(s)|s' - s|.
\]

Hence,

\[
||x(s')||_{\delta_{s'}} \leq ||x(s)||_{\delta_s} + 2c(s)|s' - s|,
\]

and

\[
||x(s)||_{\delta_s} \leq ||x(s')||_{\delta_{s'}} + c(s')|s' - s|
\]

is similar.

**Remark 1** In this case of a regular couple of nontrivial pseudolattices, the estimate of Corollary 1 for the commutator of the translation mappings becomes

\[
||[T, \mathcal{R}]||_{\bar{X}_{E,s}, \bar{X}_{E,s'}} \leq 2c(s, s')|s' - s||T||,
\]

as in (3.3) of [12].

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References


