SAWYER-TYPE INEQUALITIES FOR LORENTZ SPACES

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Abstract. The Hardy-Littlewood maximal operator satisfies the classical Sawyer-type estimate

$$\left\| \frac{Mf}{v} \right\|_{L^{1,\infty}(uv)} \leq C_{u,v} \|f\|_{L^1(u)},$$

where $u \in A_1$ and $uv \in A_\infty$. We prove a novel extension of this result to the general restricted weak type case. That is, for $p > 1$, $u \in A_p^{\mathcal{R}}$, and $uv^p \in A_\infty$,

$$\left\| \frac{Mf}{v} \right\|_{L^{p,\infty}(uv^p)} \leq C_{u,v} \|f\|_{L^{p,1}(u)}.$$

From these estimates, we deduce new weighted restricted weak type bounds and Sawyer-type inequalities for the m-fold product of Hardy-Littlewood maximal operators. We also present an innovative technique that allows us to transfer such estimates to a large class of multi-variable operators, including m-linear Calderón-Zygmund operators, avoiding the A_∞ extrapolation theorem and producing many estimates that have not appeared in the literature before. Our results combine Sawyer-type inequalities, $A_p^{\mathcal{R}}$ weights, and Lorentz spaces.

1. Introduction

"Sawyer-type inequalities" is a terminology coined in the paper [CUMP2], where the authors prove that if $u \in A_1$, and $v \in A_1$ or $uv \in A_\infty$, then

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{|T(fv)(x)|}{v(x)} > t\right\}\right) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|u(x)v(x)dx, \quad t > 0, \tag{1.1}$$

where T is either the Hardy-Littlewood maximal operator or a linear Calderón-Zygmund operator. This result extends some questions previously considered by B. Muckenhoupt and R. Wheeden in [MW], and solves in the affirmative a conjecture formulated by E. Sawyer in [S] concerning the Hilbert transform. These problems were advertised by B. Muckenhoupt in [M3], where the terminology "mixed type norm inequalities" was introduced and was also used since then in other papers like [AM] or [MROS]. In general, this terminology refers to certain weighted estimates for some classical operators T, where a weight v is included in their level sets, that is,

$$\left\{x \in \mathbb{R}^n : \frac{|Tf(x)|}{v(x)} > t\right\}, \quad t > 0. \tag{1.2}$$

The structure of such sets makes impossible, or very difficult, to use classical tools to measure them, such as the Vitali covering lemma or interpolation theorems.

In this paper we consider mixed restricted weak type norm inequalities, or Sawyer-type inequalities for Lorentz spaces, that is, we study estimates of the form

$$w\left(\left\{x \in \mathbb{R}^n : \frac{|Tf(x)|}{v(x)} > t\right\}\right)^{1/p} \leq \frac{C}{t} \|f\|_{L^{p,1}(u)}, \quad t > 0, \tag{1.3}$$

where $p \geq 1$, T is a classical operator, and u, v, w are weights. We also consider extensions of such inequalities to the multi-variable setting. Our goal is to prove estimates like (1.3) for sub-linear and multi-sub-linear maximal operators, and multi-linear Calderón-Zygmund operators. Observe that in the classical situation, namely when $u = w$, $v \approx 1$, and T is either the Hardy-Littlewood maximal operator or a linear Calderón-Zygmund operator, the inequality (1.3) holds if $w \in A_p^{\mathcal{R}}$ (some authors use the notation $A_{p,1}$ for this class of weights, as in [CHK]). The case when $v \not\approx 1$ is much more difficult, and in what follows, we will study it in great detail.

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The starting point of this paper and our primary motivation to consider Sawyer-type inequalities for Lorentz spaces comes from the study of the m-fold product of Hardy-Littlewood maximal operators,

$$M^{\otimes}(\vec{f})(x) := \prod_{i=1}^{m} Mf_i(x), \quad x \in \mathbb{R}^n.$$

M. J. Carro and the second author proved in [CR] that for exponents $1 \le p_1, \dots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and weights $w_1, \dots, w_m$ in A_∞ and $\nu_{\vec{w}} := w_1^{p/p_1} \dots w_m^{p/p_m}$, a necessary condition to have

$$M^{\otimes} : L^{p_1,1}(w_1) \times \cdots \times L^{p_m,1}(w_m) \longrightarrow L^{p,\infty}(\nu_{\vec{w}}) \tag{1.4}$$

is that $w_i \in A_{p_i}^{\mathcal{R}}$, for $i = 1, \dots, m$. They left as an open question to prove that this last condition is also sufficient for (1.4) to hold. It is reasonable to think that this may indeed be true since the endpoint case was proved in [LOPTTG]. That is, for weights $w_1, \dots, w_m \in A_1$, we have that

$$M^{\otimes} : L^1(w_1) \times \cdots \times L^1(w_m) \longrightarrow L^{1/m,\infty}(w_1^{1/m} \dots w_m^{1/m}). \tag{1.5}$$

To prove this result, one has to control the following quantity for $t > 0$, which is related to the level sets (1.2):

$$\nu_{\vec{w}}\left(\left\{x \in \mathbb{R}^n : M^{\otimes}(\vec{f})(x) > t\right\}\right) = \nu_{\vec{w}}\left(\left\{x \in \mathbb{R}^n : Mf_i(x) > \frac{t}{\prod_{j\neq i} Mf_j(x)}\right\}\right),$$

where $\nu_{\vec{w}} = w_1^{1/m} \dots w_m^{1/m}$. This is achieved by applying the classical Sawyer-type inequality (1.1) for the Hardy-Littlewood maximal operator M in combination with the observation that for locally integrable functions $h_1, \dots, h_k$, $\prod_{j=1}^k (Mh_j)^{-1} \in RH_\infty$, with constant depending only on k and the dimension n.

As we will show in Theorem 4.1, it turns out that the bound (1.4) holds if $w_i \in A_{p_i}^{\mathcal{R}}$, for $i = 1, \dots, m$, solving in the affirmative the open question in [CR] and completing the characterization of the restricted weak type bounds of $M^{\otimes}$ for A_∞ weights. The strategy that we follow is similar to the one in [LOPTTG] for the endpoint case (1.5), but we have to replace the classical Sawyer-type inequality (1.1) by the estimate obtained in Theorem 3.8, which is a new restricted weak Sawyer-type inequality involving the class of weights $A_p^{\mathcal{R}}$. That is,

$$\left\| \frac{Mf}{v} \right\|_{L^{p,\infty}(uv^p)} \le C_{u,v} \|f\|_{L^{p,1}(u)}, \tag{1.6}$$

for $p > 1$, $u \in A_p^{\mathcal{R}}$, and $uv^p \in A_\infty$. The $A_p^{\mathcal{R}}$ condition on the weight u is a natural assumption since it is necessary when $v \approx 1$. In Lemma 3.10 we also manage to track the dependence of the constant $C_{u,v}$ on the weights u and uv^p, even in the endpoint case $p = 1$, refining the bound (1.1) in [CUMP2].

There is no reason to restrict ourselves to the study of one-variable Sawyer-type inequalities. Quite recently, the bound (1.1) has been extended to the multi-variable setting in [LOPi]. More precisely, for weights $w_1, \dots, w_m \in A_1$, and $v \in A_\infty$,

$$\left\| \frac{\mathcal{M}(\vec{f})}{v} \right\|_{L^{1/m,\infty}(\nu_{\vec{w}} v^{1/m})} \le \left\| \frac{\prod_{i=1}^m Mf_i}{v} \right\|_{L^{1/m,\infty}(\nu_{\vec{w}} v^{1/m})} \lesssim \prod_{i=1}^m \|f_i\|_{L^1(w_i)}. \tag{1.7}$$

Inspired by this result, we follow a similar approach to extend our Sawyer-type inequality (1.6) to the multi-variable setting, obtaining a generalization of (1.7) in Theorem 4.2. That is, for weights $w_1, \dots, w_m$ and v such that for $i = 1, \dots, m$, $w_i \in A_{p_i}^{\mathcal{R}}$ and $w_i v^{p_i} \in A_\infty$,

$$\left\| \frac{\mathcal{M}(\vec{f})}{v} \right\|_{L^{p,\infty}(\nu_{\vec{w}} v^p)} \le \left\| \frac{\prod_{i=1}^m Mf_i}{v} \right\|_{L^{p,\infty}(\nu_{\vec{w}} v^p)} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(w_i)}. \tag{1.8}$$

Observe that this result is an extension of (1.4). To our knowledge, this multi-variable mixed restricted weak type inequalities for maximal operators involving the $A_p^{\mathcal{R}}$ condition on the weights have not been previously studied, and we found no record of them being conjectured in the literature.

Motivated by the conjecture of E. Sawyer in [S], we can ask ourselves if it is possible to obtain bounds like (1.8) for multi-linear Calderón-Zygmund operators T. Once again, the endpoint case

$p_1 = \cdots = p_m = 1$ has already been considered and extensively investigated in [LOPi]. There, it was shown that for weights $w_1, \dots, w_m \in A_1$, and $\nu_{\vec{w}} v^{1/m} \in A_\infty$,

$$\left\| \frac{T(\vec{f})}{v} \right\|_{L^{1/m,\infty}(\nu_{\vec{w}} v^{1/m})} \lesssim \prod_{i=1}^m \|f_i\|_{L^1(w_i)}, \tag{1.9}$$

as a corollary of (1.7), combined with a result in [OP], that allows replacing $\mathcal{M}$ by T using an extrapolation type argument based on the A_∞ extrapolation theorem obtained in [CUMP1, CGCMP]. We succeed in our goal and manage to obtain an extension of (1.9) to the general restricted weak setting. In Theorem 4.6 we prove, among other things, that for weights $w_1, \dots, w_m$ and v such that for $i = 1, \dots, m$, $w_i \in A_{p_i}^{\mathcal{R}}$ and $w_i v^{p_i} \in A_\infty$, and some other technical hypotheses on the weights,

$$\left\| \frac{T(\vec{f})}{v} \right\|_{L^{p,\infty}(\nu_{\vec{w}} v^p)} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(w_i)}. \tag{1.10}$$

To achieve this, we build upon (1.8), but unlike in [LOPi], we manage to avoid the use of extrapolation arguments like the ones in [OP]. Instead, we present in Theorem 4.4 a novel technique that allows us to replace $\mathcal{M}$ by T exploiting the fine structure of the Lorentz space $L^{p,\infty}$, the $A_p^{\mathcal{R}}$ condition, and the recent advances in sparse domination. It is worth mentioning that we couldn't find in the literature any trace of results like (1.10), involving multi-linear Calderón-Zygmund operators, $A_p^{\mathcal{R}}$ weights and mixed restricted weak type inequalities.

It is curious that we didn't find much about Sawyer-type inequalities for Lorentz spaces apart from the endpoint results studied in [CUMP2, LOP, LOPi, MW, OP, S]. As we have seen before, these inequalities are fundamental to understand the behavior of the operator $M^{\otimes}$, but they appear naturally in the study of other classical operators, even in the one-variable case. Consider, for example, the case of the Hilbert transform H. Indeed, if $p > 1$ and $w \in A_p^{\mathcal{R}}$, it is well known that $H : L^{p,1}(w) \longrightarrow L^{p,\infty}(w)$. Hence, duality, linearity and self-adjointness of H yield

$$\left\| \frac{H(fw)}{w} \right\|_{L^{p',\infty}(w)} \leq C_w \|f\|_{L^{p',1}(w)}.$$

This is an example of an estimate like (1.3) involving the $A_p^{\mathcal{R}}$ condition on the weights and obtained almost without effort. The same inequality holds for the Hardy-Littlewood maximal operator M, but we cannot use the same argument, as shown in [CS]. In Theorem 4.8 we will generalize such result for M, obtaining as a particular case an alternative proof of the result in [CS]. Sawyer-type problems also arise in the broadly studied topic involving commutators of linear operators T with a BMO function b, although we will not deal with them in this paper. The crucial initial observation is that we can write $[b, T]$ as a complex integral operator using Cauchy's integral theorem, obtaining that for $\varepsilon > 0$,

$$[b, T]f = \frac{1}{2\pi i} \int_{\{z \in \mathbb{C}:\, |z| = \varepsilon\}} \frac{T_z(f)}{z^2}\, dz,$$

where

$$T_z(f) := e^{zb} T\left(\frac{f}{e^{zb}}\right), \quad z \in \mathbb{C}.$$

This approach was introduced in the celebrated paper [CRW] and was further developed in [AKMP]. In the context of Lorentz spaces, for $p > 1$ and a weight w, and in virtue of Minkowski's inequality, we get that for any $\varepsilon > 0$,

$$\|[b, T]f\|_{L^{p,\infty}(w)} \leq \frac{1}{\varepsilon} \sup_{z \in \mathbb{C}:\, |z| = \varepsilon} \|T_z(f)\|_{L^{p,\infty}(w)}.$$

Since $b \in BMO$, as a consequence of the John-Nirenberg inequality, there is a constant $s_0 > 0$ such that for $|z| \leq s_0$, $v^{-1} := |e^{zb}| = e^{\Re(z) b} \in A_p$, and hence, it is possible to deduce weighted inequalities for commutators from estimates of the form

$$\left\| \frac{T(fv)}{v} \right\|_{L^{p,\infty}(w)} \lesssim \|f\|_X,$$

for a norm or a quasi-norm $\|\cdot\|_X$ and $v^{-1} \in A_p$. Further results for commutators involving Sawyer-type inequalities can be found in [B, BCP].

2. Preliminaries

2.1. **Lorentz spaces and classical weights.** Let us recall the definition of the Lebesgue and Lorentz spaces (see [BS]). For $p>0$ and an arbitrary measure space (X,ν), $L^{p,1}(\nu)$ is the Lorentz space of ν-measurable functions such that

$$||f||_{L^{p,1}(\nu)} := p\int_0^\infty \lambda_f^\nu(y)^{1/p}dy = \int_0^\infty f_\nu^*(t)t^{1/p}\frac{dt}{t} < \infty,$$

$L^p(\nu)$ is the Lebesgue space of ν-measurable functions such that

$$||f||_{L^p(\nu)} := \left(\int_X |f|^p \nu\right)^{1/p} < \infty,$$

and $L^{p,\infty}(\nu)$ is the Lorentz space of ν-measurable functions such that

$$||f||_{L^{p,\infty}(\nu)} := \sup_{y>0} y\lambda_f^\nu(y)^{1/p} = \sup_{t>0} t^{1/p} f_\nu^*(t) < \infty,$$

where f_ν^* is the decreasing rearrangement of f with respect to ν, defined by

$$f_\nu^*(t) := \inf\{y>0 : \lambda_f^\nu(y)\le t\}, \qquad \lambda_f^\nu(t) := \nu(\{x\in X : |f(x)|>t\}).$$

If $p\ge 1$, then $L^{p,1}(\nu)\hookrightarrow L^p(\nu)\hookrightarrow L^{p,\infty}(\nu)$. Given a σ-finite measure space (X,ν), and parameters $0<r<p<\infty$, the quantity

$$|||f|||_{L^{p,\infty}(\nu)} := \sup_{0<\nu(E)<\infty} \nu(E)^{\frac{1}{p}-\frac{1}{r}}\left(\int_E |f|^r d\nu\right)^{1/r}$$

satisfies that

$$\|f\|_{L^{p,\infty}(\nu)} \le |||f|||_{L^{p,\infty}(\nu)} \le \left(\frac{p}{p-r}\right)^{1/r}\|f\|_{L^{p,\infty}(\nu)}.$$

This result is classical (see [G, Exercise 1.1.12]), and throughout this paper, we will refer to these inequalities as Kolmogorov's inequalities.

Given $f\in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|dy,$$

where the supremum is taken over all cubes $Q\subseteq\mathbb{R}^n$ containing x. Muckenhoupt studied the boundedness of M on Lebesgue spaces $L^p(w)$ (see [M1]). Given a positive and locally integrable function w, called weight, and $1<p<\infty$,

$$M : L^p(w)\longrightarrow L^p(w)$$

if, and only if $w\in A_p$, that is, if

$$[w]_{A_p} := \sup_Q\left(\fint_Q w\right)\left(\fint_Q w^{1-p'}\right)^{p-1} < \infty,$$

where we use the notation $\fint_Q w = \frac{1}{|Q|}\int_Q w(x)dx$. Moreover, if $1\le p<\infty$,

$$M : L^p(w)\longrightarrow L^{p,\infty}(w)$$

if, and only if $w\in A_p$, where a weight $w\in A_1$ if

$$[w]_{A_1} := \sup_Q\left(\fint_Q w\right)\|\chi_Q w^{-1}\|_{L^\infty(w)} = \sup_Q\left(\fint_Q w\right)(\operatorname*{ess\,inf}_{x\in Q} w(x))^{-1} < \infty.$$

The restricted weak type bounds for M were studied in [CHK, KT]. For $1\le p<\infty$,

$$M : L^{p,1}(w)\longrightarrow L^{p,\infty}(w)$$

if, and only if $w\in A_p^{\mathcal{R}}$, where a weight w is in $A_p^{\mathcal{R}}$ (also denoted by $A_{p,1}$) if

$$[w]_{A_p^{\mathcal{R}}} := \sup_Q w(Q)^{1/p}\frac{\|\chi_Q w^{-1}\|_{L^{p',\infty}(w)}}{|Q|} < \infty,$$

or equivalently, if

$$\|w\|_{A_p^{\mathcal{R}}} := \sup_Q\sup_{E\subseteq Q}\frac{|E|}{|Q|}\left(\frac{w(Q)}{w(E)}\right)^{1/p} < \infty.$$

We have that $[w]_{A_p^{\mathcal{R}}} \le \|w\|_{A_p^{\mathcal{R}}} \le p[w]_{A_p^{\mathcal{R}}}$. Given a measurable set F, we write $w(F) = \int_F w(x)dx$. If $w = 1$, we simply write $|F|$. Moreover,

$$\|M\|_{L^{p,1}(w)\to L^{p,\infty}(w)} \approx [w]_{A_p^{\mathcal{R}}}.$$

As usual, we write $A \lesssim B$ if there exists a positive constant $C > 0$, independent of A and B, such that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

We now give the definitions of some other classes of weights that will appear later. For more information about all these, see [CUMP2, CUN, DMRO, GCRF]. Define the class of weights

$$A_\infty := \bigcup_{p\ge1} A_p = \bigcup_{p\ge1} A_p^{\mathcal{R}}.$$

A weight $w \in A_\infty$ if, and only if

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q) < \infty.$$

This quantity is usually referred to as the Fujii-Wilson A_∞ constant. More generally, given a weight u, and $p > 1$, we say that $w \in A_p(u)$ if

$$[w]_{A_p(u)} := \sup_Q \left(\frac{1}{u(Q)}\int_Q wu\right)\left(\frac{1}{u(Q)}\int_Q w^{1-p'}u\right)^{p-1} < \infty,$$

and $w \in A_1(u)$ if

$$[w]_{A_1(u)} := \sup_Q \left(\frac{1}{u(Q)}\int_Q wu\right)\|\chi_Q w^{-1}\|_{L^\infty(wu)} = \sup_Q \left(\frac{1}{u(Q)}\int_Q wu\right)(\operatorname*{ess\,inf}_{x\in Q} w(x))^{-1} < \infty,$$

and as before, we define

$$A_\infty(u) := \bigcup_{p\ge1} A_p(u).$$

If u is a doubling weight for cubes in $\mathbb{R}^n$, and $w \in A_\infty(u)$, then

$$[w]_{A_\infty(u)} := \sup_Q \frac{1}{wu(Q)} \int_Q M_u(w\chi_Q)u < \infty,$$

where

$$M_u f(x) := \sup_{Q\ni x} \frac{1}{u(Q)} \int_Q |f(y)|u(y)dy$$

is the weighted Hardy-Littlewood maximal operator. If $p > 1$, then M_u is bounded on $L^p(wu)$ if, and only if $w \in A_p(u)$, provided that u is doubling. Given $s > 1$, we say that a weight $w \in RH_s$ if

$$[w]_{RH_s} := \sup_Q \frac{|Q|}{w(Q)} \left(\fint_Q w^s\right)^{1/s} < \infty,$$

and $w \in RH_\infty$ if

$$[w]_{RH_\infty} := \sup_Q \frac{|Q|}{w(Q)} \|\chi_Q w\|_{L^\infty(\mathbb{R}^n)} = \sup_Q \frac{|Q|}{w(Q)} \operatorname*{ess\,sup}_{x\in Q} w(x) < \infty.$$

We have that

$$A_\infty = \bigcup_{1<s\le\infty} RH_s.$$

In [LOPTTG], the following multi-variable extension of the Hardy-Littlewood maximal operator was introduced in connection with the theory of multi-linear Calderón-Zygmund operators:

$$\mathcal{M}(\vec{f}) := \sup_Q \prod_{i=1}^m \left(\fint_Q |f_i|\right)\chi_Q,$$

for $\vec{f} = (f_1, \dots, f_m)$, with $f_i \in L^1_{loc}(\mathbb{R}^n)$, $i = 1, \dots, m$. For $1 \le p_1, \dots, p_m < \infty$, $\vec{P} := (p_1, \dots, p_m)$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and weights $w_1, \dots, w_m$, with $\vec{w} := (w_1, \dots, w_m)$, $\nu_{\vec{w}} := w_1^{p/p_1} \dots w_m^{p/p_m}$,

$$\mathcal{M} : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \longrightarrow L^{p,\infty}(\nu_{\vec{w}})$$

if, and only if $\vec{w} \in A_{\vec{P}}$, that is, if

$$[\vec{w}]_{A_{\vec{P}}} := \sup_Q \left(\fint_Q \nu_{\vec{w}}\right)^{1/p} \prod_{i=1}^m \left(\fint_Q w_i^{1-p_i'}\right)^{1/p_i'} < \infty,$$

where $\left(\fint_Q w_i^{1-p_i'}\right)^{1/p_i'}$ is replaced by $(\operatorname{ess\,inf}_{x\in Q} w_i(x))^{-1}$ if $p_i = 1$. Moreover, if $1 < p_1, \dots, p_m < \infty$, then

$$\mathcal{M} : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \longrightarrow L^p(\nu_{\vec{w}})$$

if, and only if $\vec{w} \in A_{\vec{P}}$.

We are using throughout the paper the standard notation $T : X_1 \times \cdots \times X_m \longrightarrow X_0$ to denote that T is a bounded operator from $X_1 \times \cdots \times X_m$ to X_0, where X_i is an appropriate function space.

2.2. **Dyadic grids and sparse collections of cubes.** A general dyadic grid $\mathscr{D}$ is a collection of cubes in $\mathbb{R}^n$ with the following properties:

1) For any $Q \in \mathscr{D}$, its side length l_Q is of the form 2^k, for some $k \in \mathbb{Z}$.
2) For all $Q, R \in \mathscr{D}$, $Q \cap R \in \{\emptyset, Q, R\}$.
3) The cubes of a fixed side length 2^k form a partition of $\mathbb{R}^n$.

The standard dyadic grid in $\mathbb{R}^n$ consists of the cubes $2^{-k}([0,1)^n + j)$, with $k \in \mathbb{Z}$ and $j \in \mathbb{Z}^n$. It is well known (see [HP]) that if one considers the perturbed dyadic grids

$$\mathscr{D}_\alpha := \{2^{-k}([0,1)^n + j + \alpha) : k \in \mathbb{Z}, j \in \mathbb{Z}^n\},$$

with $\alpha \in \{0, 1/3\}^n$, then for any cube $Q \subset \mathbb{R}^n$, there exist α and a cube $Q_\alpha \in \mathscr{D}_\alpha$ such that $Q \subseteq Q_\alpha$ and $l_{Q_\alpha} \leq 6 l_Q$.

A collection of cubes $\mathcal{S}$ is said to be η-sparse if there exists $0 < \eta < 1$ such that for every cube $Q \in \mathcal{S}$, there exists a set $E_Q \subseteq Q$ with $\eta|Q| \leq |E_Q|$, and for every $Q \neq R \in \mathcal{S}$, $E_R \cap E_Q = \emptyset$.

2.3. **Calderón-Zygmund operators.** We say that a function $\omega : [0, \infty) \to [0, \infty)$ is a modulus of continuity if it is continuous, increasing, sub-additive and such that $\omega(0) = 0$. We say that ω satisfies the Dini condition if

$$\|\omega\|_{\mathrm{Dini}} := \int_0^1 \frac{\omega(t)}{t} dt < \infty.$$

We give the definition of the multi-linear ω-Calderón-Zygmund operators. We denote by $\mathscr{S}(\mathbb{R}^n)$ the space of all Schwartz functions on $\mathbb{R}^n$ and by $\mathscr{S}'(\mathbb{R}^n)$ its dual space, the set of all tempered distributions on $\mathbb{R}^n$.

Definition 2.1. An m-linear ω-Calderón-Zygmund operator is an m-linear and continuous operator $T : \mathscr{S}(\mathbb{R}^n) \times \cdots \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ for which there exists a locally integrable function $K(y_0, y_1, \dots, y_m)$ defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ satisfying the size estimate

$$|K(y_0, y_1, \dots, y_m)| \leq \frac{C_K}{(|y_0 - y_1| + \cdots + |y_0 - y_m|)^{nm}}$$

for all $(y_0, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $y_0 \neq y_j$ for some $j \in \{1, \dots, m\}$, and for every $i = 0, \dots, m$, the smoothness estimate

$$\begin{aligned} &|K(y_0, y_1, \dots, y_i, \dots, y_m) - K(y_0, y_1, \dots, y_i', \dots, y_m)| \\ &\qquad \leq \frac{C_K}{(|y_0 - y_1| + \cdots + |y_0 - y_m|)^{nm}} \omega\left(\frac{|y_i - y_i'|}{(|y_0 - y_1| + \cdots + |y_0 - y_m|)^{nm}}\right), \end{aligned}$$

whenever $|y_i - y_i'| \leq \frac{1}{2} \max_{1\leq j\leq m} |y_0 - y_j|$ and for some constant $C_K > 0$, such that

$$T(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m) dy_1 \dots dy_m,$$

whenever $f_1, \dots, f_m \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ and $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$, and for some $1 \leq q_1, \dots, q_m < \infty$, T extends to a bounded m-linear operator from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, with $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$.

If we take $\omega(t) = t^\varepsilon$ for some $\varepsilon > 0$, we recover the classical multi-linear Calderón-Zygmund operators. In general, an m-linear ω-Calderón-Zygmund operator with ω satisfying the Dini condition can be extended to a bounded operator from $L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n)$ to $L^{1/m,\infty}(\mathbb{R}^n)$. The multilinear Calderón-Zygmund theory has been investigated by many authors. For more information on this matter, see [GT, LOPTTG, LZ] and the publications cited there.

3. Sawyer-type inequalities for maximal operators

We devote this section to the study of a novel restricted weak type inequality that extends the classical Sawyer-type inequality (1.1) for the Hardy-Littlewood maximal operator. To this end, we will need some previous results.

The following lemma contains well known results on weights (see [CUMP2, CUN, GCRF, LOPi]), but we will give most of their proofs since we need to keep track of the constants of the weights involved.

Lemma 3.1. *Let u and w be weights.*

(a) *If $u \in A_1$, then $u^{-1} \in RH_\infty$, and $[u^{-1}]_{RH_\infty} \leq [u]_{A_1}$.*
(b) *If $u \in RH_\infty$, and $q \geq 1$, then $u^q \in RH_\infty$, and $[u^q]_{RH_\infty} \leq [u]^q_{RH_\infty}$.*
(c) *If $u \in RH_\infty$, and $[u]_{RH_\infty} \leq \beta$, then there exists $r > 1$, depending only on n, β, such that $u \in A_r$ and $[u]_{A_r} \leq c_{n,\beta}$. In particular, $RH_\infty \subseteq A_\infty$.*
(d) *If $u \in A_\infty$, and $w \in RH_\infty$, then $uw \in A_\infty$.*
(e) *If $u \in A_1 \cap RH_\infty$, then $u \approx 1$.*

Fix $p \geq 1$, and $f_1, \dots, f_m \in L^1_{loc}(\mathbb{R}^n)$, and let $v := \prod_{i=1}^m (Mf_i)^{-1}$.

(f) *$v^p \in RH_\infty$, and $1 \leq [v^p]_{RH_\infty} \leq c_{m,n,p}$.*
(g) *If $u \in A_\infty$, then $uv^p \in A_\infty$, with constant independent of $\vec{f} = (f_1, \dots, f_m)$.*

Proof. To prove (a), fix a cube $Q \subseteq \mathbb{R}^n$. By Hölder's inequality, we have that

$$|Q| = \int_Q u^{-1/2} u^{1/2} \leq \left(\int_Q u^{-1} \right)^{1/2} \left(\int_Q u \right)^{1/2},$$

and hence,

$$\operatorname*{ess\,sup}_{x \in Q} u(x)^{-1} = (\operatorname*{ess\,inf}_{x \in Q} u(x))^{-1} \leq [u]_{A_1} \frac{|Q|}{u(Q)} \leq [u]_{A_1} \fint_Q u^{-1},$$

and the desired result follows taking the supremum over all cubes Q.

To prove (b), fix a cube $Q \subseteq \mathbb{R}^n$. Then,

$$\operatorname*{ess\,sup}_{x \in Q} u(x) \leq [u]_{RH_\infty} \fint_Q u \leq [u]_{RH_\infty} \left(\fint_Q u^q \right)^{1/q},$$

from which the desired result follows, as before.

To prove (c), fix a cube $Q \subseteq \mathbb{R}^n$, and a measurable set $E \subseteq Q$. Then,

$$\frac{u(E)}{u(Q)} = \frac{1}{u(Q)} \int_Q \chi_E u \leq \frac{|E|}{u(Q)} \operatorname*{ess\,sup}_{x \in Q} u(x) \leq [u]_{RH_\infty} \frac{|E|}{|Q|} \leq \beta \frac{|E|}{|Q|}.$$

In particular, for every $\varepsilon > 0$, and $\delta = \varepsilon/\beta$, if $|E| < \delta|Q|$, then $u(E) < \varepsilon u(Q)$, and the desired result follows from this fact applying the last theorem in [M2].

To prove (d), take $q, r > 1$ such that $u \in A_q$ and $w \in A_r$. We will show that $uw \in A_s$, for $s = q + r - 1$. Fix a cube $Q \subseteq \mathbb{R}^n$. Then,

$$\fint_Q uw \leq [w]_{RH_\infty} \left(\fint_Q u \right) \left(\fint_Q w \right),$$

and in virtue of Hölder's inequality with exponent $\alpha = 1 + \frac{r-1}{q-1}$,

$$\begin{aligned}\left(\fint_Q (uw)^{1-s'} \right)^{s-1} &\leq \left(\fint_Q u^{(1-s')\alpha} \right)^{(s-1)/\alpha} \left(\fint_Q w^{(1-s')\alpha'} \right)^{(s-1)/\alpha'} \\ &= \left(\fint_Q u^{1-q'} \right)^{q-1} \left(\fint_Q w^{1-r'} \right)^{r-1},\end{aligned}$$

so $[uw]_{A_s} \leq [w]_{RH_\infty} [u]_{A_q} [w]_{A_r} < \infty$.

The property (e) follows immediately from Corollary 4.6 in [CUN].

To prove (f), observe that, in virtue of [G, Theorem 7.2.7], we have that for $0 < \delta < 1$, $(Mf_i)^\delta \in A_1$, and $[(Mf_i)^\delta]_{A_1} \leq \frac{c_n}{1-\delta}$, $i = 1, \dots, m$. In particular, $w := \prod_{i=1}^m (Mf_i)^{\delta/m} \in A_1$, and $[w]_{A_1} \leq \prod_{i=1}^m [(Mf_i)^\delta]^{1/m}_{A_1} \leq \frac{c_n}{1-\delta}$. Since $v^p = w^{-mp/\delta}$, it follows from (a) and (b) that

$$[v^p]_{RH_\infty} \leq [w^{-1}]^{mp/\delta}_{RH_\infty} \leq [w]^{mp/\delta}_{A_1} \leq \left(\frac{c_n}{1-\delta} \right)^{mp/\delta},$$

so

$$1 \leq [v^p]_{RH_\infty} \leq c_{m,n,p} := \inf_{0<\delta<1} \left(\frac{c_n}{1-\delta}\right)^{mp/\delta}.$$

To prove (g), we already know by (f) that $v^p \in RH_\infty$, with constant bounded by $c_{m,n,p}$, so by (c), there exists $r > 1$, depending only on m, n, p, such that $[v^p]_{A_r} \leq C_{m,n,p}$. By (d), for $q > 1$ such that $u \in A_q$, and $s = s(m,n,p,q) = q + r - 1$, $[uv^p]_{A_s} \leq C'_{m,n,p}[u]_{A_q} < \infty$. □

The next lemma gives a result on weights that will be handy later on.

Lemma 3.2. *Let u and v be weights, and suppose that $u \in A_\infty$. Then, $uv \in A_\infty$ if, and only if $v \in A_\infty(u)$.*

Proof. Let us first assume that $uv \in A_\infty$. Since $u \in A_\infty$, there exists $s > 1$ such that $u \in RH_s$, and since $uv \in A_\infty$, there exists $r > 1$ such that $uv \in A_r$. Take $q := \frac{rs}{s-1} > 1$. We will show that $v \in A_q(u)$. Fix a cube Q. Then,

$$I_Q := \left(\frac{1}{u(Q)}\int_Q vu\right)\left(\frac{1}{u(Q)}\int_Q v^{1-q'}u\right)^{q-1} = \left(\frac{|Q|}{u(Q)}\right)^q \left(\frac{1}{|Q|}\int_Q vu\right)\left(\frac{1}{|Q|}\int_Q (vu)^{1-q'}u^{q'}\right)^{q-1}.$$

Take $\alpha := \frac{q-1}{r-1} = 1 + \frac{r}{(r-1)(s-1)} > 1$ and observe that $(1-q')\alpha = 1 - r'$, $\frac{q-1}{\alpha} = r - 1$, $q'\alpha' = s$ and $\frac{q-1}{\alpha'} = \frac{q}{s}$. Using Hölder's inequality with exponent α, we get that

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q (vu)^{1-q'}u^{q'}\right)^{q-1} &\leq \left(\frac{1}{|Q|}\int_Q (vu)^{(1-q')\alpha}\right)^{(q-1)/\alpha}\left(\frac{1}{|Q|}\int_Q u^{q'\alpha'}\right)^{(q-1)/\alpha'} \\ &= \left(\frac{1}{|Q|}\int_Q (vu)^{1-r'}\right)^{r-1}\left(\frac{1}{|Q|}\int_Q u^s\right)^{q/s} \\ &\leq [u]^q_{RH_s}\left(\frac{1}{|Q|}\int_Q (vu)^{1-r'}\right)^{r-1}\left(\frac{u(Q)}{|Q|}\right)^q.\end{aligned}$$

Hence,

$$I_Q \leq [u]^q_{RH_s}\left(\frac{1}{|Q|}\int_Q vu\right)\left(\frac{1}{|Q|}\int_Q (vu)^{1-r'}\right)^{r-1} \leq [u]^q_{RH_s}[uv]_{A_r},$$

and $[v]_{A_q(u)} = \sup_Q I_Q \leq [u]^q_{RH_s}[uv]_{A_r} < \infty$.

For the converse, let us assume that $v \in A_\infty(u)$. It follows from Theorem 3.1 in [DMRO] that there exist $\delta, C > 0$ such that for every cube $Q \subseteq \mathbb{R}^n$ and every measurable set $E \subseteq Q$,

$$\frac{u(E)}{u(Q)} \leq C\left(\frac{uv(E)}{uv(Q)}\right)^\delta.$$

Similarly, since $u \in A_\infty$, there exist $\varepsilon, c > 0$ such that for every cube $Q \subseteq \mathbb{R}^n$ and every measurable set $E \subseteq Q$,

$$\frac{|E|}{|Q|} \leq c\left(\frac{u(E)}{u(Q)}\right)^\varepsilon,$$

so for every cube $Q \subseteq \mathbb{R}^n$ and every measurable set $E \subseteq Q$,

$$\frac{|E|}{|Q|} \leq cC^\varepsilon\left(\frac{uv(E)}{uv(Q)}\right)^{\varepsilon\delta},$$

and hence, $uv \in A_\infty$. □

Remark 3.3. This result is an extension of Lemma 2.1 in [CUMP2], where it is shown that if $u \in A_1$ and $v \in A_\infty(u)$, then $uv \in A_\infty$.

We introduce a weighted version of the dyadic Hardy-Littlewood maximal operator.

Definition 3.4. Let $\mathscr{D}$ be a general dyadic grid in $\mathbb{R}^n$, and let u be a weight. For a measurable function f, we consider the function

$$M_u^{\mathscr{D}} f(x) := \sup_{\mathscr{D}\ni Q\ni x} \frac{1}{u(Q)}\int_Q |f(y)|u(y)dy, \ x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \in \mathscr{D}$ that contain x. If $u = 1$, we simply write $M^{\mathscr{D}} f$.

The following bound for the operator $M_u^{\mathscr{D}}$ is essential.

Theorem 3.5. *Let $\mathscr{D}$ be a general dyadic grid in $\mathbb{R}^n$, and let u and v be weights. If $u \in A_\infty$ and $uv \in A_\infty$, then there exists a constant $C_{u,v}$, independent of $\mathscr{D}$, such that for every $t > 0$ and every measurable function f,*

$$\left\| \frac{M_u^{\mathscr{D}}(fv)}{v} \right\|_{L^{1,\infty}(uv)} \le C_{u,v} \int_{\mathbb{R}^n} |f(x)|u(x)v(x)dx.$$

Proof. In virtue of Lemma 3.2, $v \in A_\infty(u)$ and hence, this theorem follows from the proof of Theorem 1.4 in [CUMP2]. □

Remark 3.6. If we examine the proof of Theorem 1.4 in [CUMP2], and we combine it with Appendix A in [CUMP3], we can take

$$C_{u,v} = 2^q (2^n r[uv]_{A_r^{\mathcal{R}}})^{r(q-1)} \|M_u\|^q_{L^q(uv^{1-q})},$$

where $r, q > 1$ are such that $uv \in A_r^{\mathcal{R}}$ and $v \in A_{q'}(u)$.

Remark 3.7. The bound of Theorem 3.5 also holds for the weighted Hardy-Littlewood maximal operator M_u, with constant

$$C := 2^n 6^{np} p^p [u]^p_{A_p^{\mathcal{R}}} C_{u,v},$$

where $p \ge 1$ is such that $u \in A_p^{\mathcal{R}}$.

We can now state and prove the main result of this section.

Theorem 3.8. *Fix $p \ge 1$ and let u and v be weights such that $u \in A_p^{\mathcal{R}}$ and $uv^p \in A_\infty$. Then, there exists a constant C such that for every measurable function f,*

$$\left\| \frac{Mf}{v} \right\|_{L^{p,\infty}(uv^p)} \le C \|f\|_{L^{p,1}(u)}.$$

Proof. It is known (see [HP, L]) that there exists a collection $\{\mathscr{D}_\alpha\}_\alpha$ of 2^n general dyadic grids in $\mathbb{R}^n$ such that

$$Mf \le 6^n \sum_{\alpha=1}^{2^n} M^{\mathscr{D}_\alpha} f.$$

Hence,

$$\left\| \frac{Mf}{v} \right\|_{L^{p,\infty}(uv^p)} \le 12^n \sum_{\alpha=1}^{2^n} \left\| \frac{M^{\mathscr{D}_\alpha} f}{v} \right\|_{L^{p,\infty}(uv^p)},$$

and it suffices to establish the result for the operator $M^{\mathscr{D}}$, with $\mathscr{D}$ a general dyadic grid in $\mathbb{R}^n$.

We first discuss the case $p = 1$, which was proved in [CUMP2]. We reproduce the proof here keeping track of the constants. Indeed, by the definition of the A_1 condition,

$$\frac{1}{|Q|} \int_Q |f| \le [u]_{A_1} \frac{1}{u(Q)} \int_Q |f| u,$$

so we get that $M^{\mathscr{D}} f \le [u]_{A_1} M_u^{\mathscr{D}} f$. This estimate combined with Theorem 3.5 gives that

$$\left\| \frac{M^{\mathscr{D}} f}{v} \right\|_{L^{1,\infty}(uv)} \le [u]_{A_1} \left\| \frac{M_u^{\mathscr{D}}(vf/v)}{v} \right\|_{L^{1,\infty}(uv)} \le [u]_{A_1} C_{u,v} \int_{\mathbb{R}^n} |f| u,$$

and hence, the desired result follows, with $C := 24^n [u]_{A_1} C_{u,v}$.

Now, we discuss the case $p > 1$. Let us take $f = \chi_E$, with E a measurable set in $\mathbb{R}^n$, and fix a cube $Q \in \mathscr{D}$. As before, by the definition of the $A_p^{\mathcal{R}}$ condition,

$$\frac{1}{|Q|} \int_Q f \le \|u\|_{A_p^{\mathcal{R}}} \left(\frac{u(E \cap Q)}{u(Q)} \right)^{1/p},$$

so we get that $M^{\mathscr{D}}(\chi_E) \le p[u]_{A_p^{\mathcal{R}}} (M_u^{\mathscr{D}}(\chi_E))^{1/p}$. In particular,

$$\left\| \frac{M^{\mathscr{D}}(\chi_E)}{v} \right\|_{L^{p,\infty}(uv^p)} \le p[u]_{A_p^{\mathcal{R}}} \left\| \frac{M_u^{\mathscr{D}}(\chi_E)}{v^p} \right\|^{1/p}_{L^{1,\infty}(uv^p)}.$$

We can now apply Theorem 3.5 to conclude that

$$\left\| \frac{M_u^{\mathscr{D}}(\chi_E)}{v^p} \right\|_{L^{1,\infty}(uv^p)} = \left\| \frac{M_u^{\mathscr{D}}(v^p \chi_E / v^p)}{v^p} \right\|_{L^{1,\infty}(uv^p)} \le C_{u,v^p} u(E).$$

Combining all the previous estimates, we have that

$$\left\| \frac{M(\chi_E)}{v} \right\|_{L^{p,\infty}(uv^p)} \leq 24^n [u]_{A_p^{\mathcal{R}}} C_{u,v^p}^{1/p} \|\chi_E\|_{L^{p,1}(u)}.$$

Since $p > 1$, $L^{p,\infty}(uv^p)$ is a Banach space, and by standard arguments, we can extend the previous estimate to arbitrary measurable functions f, gaining a factor of $4p'$ in the constant. Hence, the desired result follows, with $C := 4 \cdot 24^n p' [u]_{A_p^{\mathcal{R}}} C_{u,v^p}^{1/p}$. □

Remark 3.9. For $p = 1$ and $u \in A_1$, a more general version of Theorem 3.8 was established in [LOP], replacing the hypothesis that $uv \in A_\infty$ by the weaker assumption that $v \in A_\infty$. It is unknown to us whether the hypothesis that $uv^p \in A_\infty$ can be replaced by $v \in A_\infty$ when $p > 1$.

In virtue of Lemma 3.1, if $u \in A_\infty$ and $v \in RH_\infty$, then for every $p \geq 1$, $uv^p \in A_\infty$, and we have a whole class of non-trivial examples of weights that satisfy the hypotheses of Theorem 3.8.

Observe that the conclusion of Theorem 3.8 is completely elementary if $p > 1$ and $u \in A_p$, since

$$\left\| \frac{Mf}{v} \right\|_{L^{p,\infty}(uv^p)} \leq \left\| \frac{Mf}{v} \right\|_{L^p(uv^p)} = \|Mf\|_{L^p(u)} \lesssim \|f\|_{L^p(u)} \lesssim \|f\|_{L^{p,1}(u)}.$$

However, this argument doesn't work in the general case, because the inequality

$$\left\| \frac{h}{v} \right\|_{L^{p,\infty}(uv^p)} \lesssim \|h\|_{L^{p,\infty}(u)}$$

may fail for some measurable functions h.

To provide applications of Theorem 3.8 we need to give a more precise estimate of the constant C that appears there in terms of the corresponding constants of the weights involved. We achieve this in the following lemma.

Lemma 3.10. *In Theorem 3.8, if $r \geq 1$ is such that $uv^p \in A_r^{\mathcal{R}}$, then one can take*

$$C = \mathscr{E}_{r,p}^n([u]_{A_p^{\mathcal{R}}}, [uv^p]_{A_r^{\mathcal{R}}}),$$

where $\mathscr{E}_{r,p}^n : [1, +\infty)^2 \longrightarrow (0, +\infty)$ is a function that increases in each variable, and it depends only on r, p, and the dimension n.

Proof. We first discuss the case when $r > 1$. We already know that we can take

$$C = \begin{cases} 24^n [u]_{A_1} C_{u,v}, & p = 1; \\ 4 \cdot 24^n p' [u]_{A_p^{\mathcal{R}}} C_{u,v^p}^{1/p}, & p > 1, \end{cases}$$

and in virtue of Remark 3.6,

$$C_{u,v^p} = 2^q (2^n r [uv^p]_{A_r^{\mathcal{R}}})^{r(q-1)} \|M_u\|_{L^q(uv^{p(1-q)})}^q,$$

where $r, q > 1$ are such that $uv^p \in A_r^{\mathcal{R}}$ and $v^p \in A_{q'}(u)$. For convenience, we write $V = v^p$. Let us first bound the factor $\|M_u\|_{L^q(uV^{1-q})}^q$. For the space of homogeneous type $(\mathbb{R}^n, d_\infty, u(x)dx)$, it follows from the proof of Theorem 1.3 in [HPR] that

$$\|M_u\|_{L^q(uV^{1-q})}^q \leq 2^{q-1} q' 40^{qD_u} (1 + 6 \cdot 800^{D_u}) [V]_{A_\infty(u)} [V]_{A_{q'}(u)}^{q-1},$$

where $D_u = p \log_2(2^n p [u]_{A_p^{\mathcal{R}}})$. Now, given a cube $Q \subseteq \mathbb{R}^n$, and applying Hölder's inequality with exponent q, we have that

$$\begin{aligned} \int_Q M_u(V\chi_Q) u &= \int_Q \frac{M_u(V\chi_Q)}{V} uV \leq \left\| \frac{M_u(V\chi_Q)}{V} \right\|_{L^q(uV)} uV(Q)^{1/q'} \\ &= \|M_u(V\chi_Q)\|_{L^q(uV^{1-q})} uV(Q)^{1/q'} \\ &\leq \|M_u\|_{L^q(uV^{1-q})} \|V\chi_Q\|_{L^q(uV^{1-q})} uV(Q)^{1/q'} \\ &= \|M_u\|_{L^q(uV^{1-q})} uV(Q), \end{aligned}$$

and taking the supremum over all cubes Q, we get that $[V]_{A_\infty(u)} \leq \|M_u\|_{L^q(uV^{1-q})}$. Combining the previous estimates, we obtain that

$$\|M_u\|_{L^q(uV^{1-q})}^q \leq (2^{q-1} q' 40^{qD_u} (1 + 6 \cdot 800^{D_u}))^{q'} [V]_{A_{q'}(u)}^q.$$

Now, we will bound the factor $[V]^q_{A_{q'}(u)}$. In virtue of [HP, Proposition 2.2], and using the definitions of $[u]_{A_{2p}}$ and $[u]_{A_p^{\mathcal{R}}}$, and Kolmogorov's inequalities, we can deduce that

$$[u]_{A_\infty} \leq c_n [u]_{A_{2p}} \leq (2p-1)^{2p-1} c_n [u]^{2p}_{A_p^{\mathcal{R}}} =: c_{p,n} [u]^{2p}_{A_p^{\mathcal{R}}},$$

and applying Theorem 2.3 in [HPR], $u \in RH_s$ for $s = 1 + \frac{1}{2^{n+1} c_{p,n} [u]^{2p}_{A_p^{\mathcal{R}}} - 1}$, and $[u]_{RH_s} \leq 2$. Since $uV \in A_{2r}$, Lemma 3.2 tells us that if we choose $q' = 2rs'$, then

$$[V]^q_{A_{q'}(u)} \leq [u]^{qq'}_{RH_s} [uV]^q_{A_{2r}} \leq 2^{qq'} (2r-1)^{q(2r-1)} [uV]^{2rq}_{A_r^{\mathcal{R}}}.$$

Finally, observe that $q' = 2^{n+2} r c_{p,n} [u]^{2p}_{A_p^{\mathcal{R}}}$, and $1 < q \leq 2$, so

$$\begin{aligned} C_{u,V} &\leq 2^2 (2^n r [uV]_{A_r^{\mathcal{R}}})^r \times (2q' 40^{2D_u} (1 + 6 \cdot 800^{D_u}))^{q'} \times 2^{2q'} (2r-1)^{4r-2} [uV]^{4r}_{A_r^{\mathcal{R}}} \\ &\leq 2^{2+nr} (2r-1)^{4r-2} r^r [uv^p]^{5r}_{A_r^{\mathcal{R}}} \left(2^{n+5} r c_{p,n} [u]^{2p}_{A_p^{\mathcal{R}}} 40^{5p \log_2 (2^n p [u]_{A_p^{\mathcal{R}}})}\right)^{2^{n+2} r c_{p,n} [u]^{2p}_{A_p^{\mathcal{R}}}} \\ &=: C^n_{r,p}([u]_{A_p^{\mathcal{R}}}, [uv^p]_{A_r^{\mathcal{R}}}), \end{aligned}$$

and the desired result follows, with

$$\mathscr{E}^n_{r,p}([u]_{A_p^{\mathcal{R}}}, [uv^p]_{A_r^{\mathcal{R}}}) = \begin{cases} 24^n [u]_{A_1} C^n_{r,1}([u]_{A_1}, [uv]_{A_r^{\mathcal{R}}}), & p = 1; \\ 4 \cdot 24^n p' [u]_{A_p^{\mathcal{R}}} C^n_{r,p}([u]_{A_p^{\mathcal{R}}}, [uv^p]_{A_r^{\mathcal{R}}})^{1/p}, & p > 1. \end{cases}$$

The case when $r = 1$ follows, for example, from the case when $r = 2$ and the fact that if $uv^p \in A_1$, then $[uv^p]_{A_2^{\mathcal{R}}} \leq [uv^p]^{1/2}_{A_2} \leq [uv^p]^{1/2}_{A_1}$. □

4. Applications

In this section, we will provide several applications of the Sawyer-type inequality established in Theorem 3.8, obtaining mixed restricted weak type estimates for multi-variable maximal operators, sparse operators and Calderón-Zygmund operators.

The first result that we present is the converse of Theorem 3.3 in [CR], that was left as an open question. Combining both theorems, we obtain the complete characterization of the restricted weak type bounds of the operator $M^{\otimes}$ for A_∞ weights.

Theorem 4.1. *Let $1 \leq p_1, \dots, p_m < \infty$, and let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let $w_1, \dots, w_m$ be weights, with $w_i \in A^{\mathcal{R}}_{p_i}$, $i = 1, \dots, m$, and write $\nu_{\vec{w}} = w_1^{p/p_1} \dots w_m^{p/p_m}$. Then, there exists a constant $C > 0$ such that the inequality*

$$\left\| M^{\otimes}(\vec{f}) \right\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(w_i)}$$

holds for every vector of measurable functions $\vec{f} = (f_1, \dots, f_m)$.

Proof. The case when $p_1 = \dots = p_m = 1$ was proved in [LOPTTG], and we build upon that proof to demonstrate the remaining cases.

We can assume, without loss of generality, that $f_i \in L^\infty_c(\mathbb{R}^n)$, $i = 1, \dots, m$. Fix $t > 0$ and define

$$E_t := \{x \in \mathbb{R}^n : t < M^{\otimes}(\vec{f})(x) \leq 2t\}.$$

For $i = 1, \dots, m$, and taking $\tilde{v}_i = \prod_{j \neq i} (Mf_j)^{-1}$, we have that $E_t = \{x \in \mathbb{R}^n : t\tilde{v}_i(x) < Mf_i(x) \leq 2t\tilde{v}_i(x)\}$. Using the fact that $\tilde{v}_i \in RH_\infty$, with constant independent of $\vec{f}$ (see Lemma 3.1), Hölder's inequality, and Theorem 3.8, we obtain that

$$\begin{aligned} \lambda^{\nu_{\vec{w}}}_{M^{\otimes}(\vec{f})}(t) - \lambda^{\nu_{\vec{w}}}_{M^{\otimes}(\vec{f})}(2t) &= \int_{E_t} \nu_{\vec{w}} \leq \int_{E_t} \left(\frac{M^{\otimes}(\vec{f})}{t} \right)^p \nu_{\vec{w}} \\ &\leq \frac{1}{t^p} \prod_{i=1}^m \left(\int_{E_t} (Mf_i)^{p_i} w_i \right)^{p/p_i} \\ &\leq 2^{mp} t^{(m-1)p} \prod_{i=1}^m \left(\int_{\left\{ \frac{Mf_i}{\tilde{v}_i} > t \right\}} \tilde{v}_i^{p_i} w_i \right)^{p/p_i} \\ &\leq 2^{mp} C_1^p \dots C_m^p \frac{1}{t^p} \prod_{i=1}^m \|f_i\|^p_{L^{p_i,1}(w_i)}. \end{aligned}$$

Iterating this result, we get that for each $t > 0$ and every natural number N,

$$\lambda^{\nu_{\vec{w}}}_{M^{\otimes}(\vec{f})}(t) \leq 2^{mp} C_1^p \dots C_m^p \left(\sum_{j=0}^{N} \frac{1}{2^{jp}} \right) \frac{1}{t^p} \prod_{i=1}^{m} \|f_i\|^p_{L^{p_i,1}(w_i)} + \lambda^{\nu_{\vec{w}}}_{M^{\otimes}(\vec{f})}(2^{N+1}t),$$

and letting N tend to infinity, the last term vanishes, and we conclude that

$$\lambda^{\nu_{\vec{w}}}_{M^{\otimes}(\vec{f})}(t) \leq \frac{2^{(m+1)p}}{2^p - 1} C_1^p \dots C_m^p \frac{1}{t^p} \prod_{i=1}^{m} \|f_i\|^p_{L^{p_i,1}(w_i)} .$$

Observe that in virtue of Lemma 3.1, for $i = 1, \dots, m$, we have that $w_i \tilde{v}_i^{p_i} \in A_{s_i}$, where $s_i > 1$ depends only on m, n, p_i, and $[w_i \tilde{v}_i^{p_i}]_{A_{s_i}} \lesssim_{m,n,p_i} [w_i]_{A_{2p_i}} \lesssim_{m,n,p_i} [w_i]^{2p_i}_{A^{\mathcal{R}}_{p_i}}$, so by Lemma 3.10, we have that $C_i \leq \mathscr{E}^n_{s_i,p_i}([w_i]_{A^{\mathcal{R}}_{p_i}}, C_{m,n,p_i}[w_i]^{2p_i}_{A^{\mathcal{R}}_{p_i}})$, and hence, the desired result follows, with

$$C := \frac{2^{m+1}}{(2^p - 1)^{1/p}} \prod_{i=1}^{m} \mathscr{E}^n_{s_i,p_i}([w_i]_{A^{\mathcal{R}}_{p_i}}, C_{m,n,p_i}[w_i]^{2p_i}_{A^{\mathcal{R}}_{p_i}}),$$

which depends on the constants of the weights w_i in an increasing way. □

The next application that we provide is an extension of Theorem 3.8 to the multi-variable setting, which in turn, extends Theorem 4.1. The proof is based on the previous one, and is similar to that of Theorem 1.4 in [LOPi].

Theorem 4.2. *Let $1 \leq p_1, \dots, p_m < \infty$, and let $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Let $w_1, \dots, w_m$ be weights, with $w_i \in A^{\mathcal{R}}_{p_i}$, $i = 1, \dots, m$, and write $\nu_{\vec{w}} = w_1^{p/p_1} \dots w_m^{p/p_m}$. Let v be a weight such that $\nu_{\vec{w}} v^p$ is a weight, and $w_i v^{p_i} \in A_\infty$, $i = 1, \dots, m$. Then, there exists a constant $C > 0$ such that the inequalities*

$$\left\| \frac{\mathcal{M}(\vec{f})}{v} \right\|_{L^{p,\infty}(\nu_{\vec{w}} v^p)} \leq \left\| \frac{M^{\otimes}(\vec{f})}{v} \right\|_{L^{p,\infty}(\nu_{\vec{w}} v^p)} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i,1}(w_i)}$$

hold for every vector of measurable functions $\vec{f} = (f_1, \dots, f_m)$.

Proof. The first inequality follows from the fact that $\mathcal{M}(\vec{f}) \leq M^{\otimes}(\vec{f})$. For the second one, we can assume, without loss of generality, that $f_i \in L^\infty_c(\mathbb{R}^n)$, $i = 1, \dots, m$. Fix $y, R > 0$ and define

$$E^R_y := \{x \in \mathbb{R}^n : |x| < R,\ yv(x) < M^{\otimes}(\vec{f})(x) \leq 2yv(x)\}.$$

For $i = 1, \dots, m$, and taking $\tilde{v}_i = \prod_{j \neq i} (Mf_j)^{-1}$, and $v_i = \tilde{v}_i v$, we have that $E^R_y = \{x \in \mathbb{R}^n : |x| < R,\ yv_i(x) < Mf_i(x) \leq 2yv_i(x)\}$. Since $\tilde{v}_i \in RH_\infty$, and $w_i v^{p_i} \in A_\infty$, we have that $w_i v_i^{p_i} \in A_\infty$, with constant independent of $\vec{f}$ (see Lemma 3.1). In virtue of Hölder's inequality and Theorem 3.8, we get that

$$\nu_{\vec{w}} v^p \left(\left\{ x \in \mathbb{R}^n : |x| < R,\ \frac{M^{\otimes}(\vec{f})(x)}{v(x)} > y \right\} \right) - \nu_{\vec{w}} v^p \left(\left\{ x \in \mathbb{R}^n : |x| < R,\ \frac{M^{\otimes}(\vec{f})(x)}{v(x)} > 2y \right\} \right)$$

$$= \int_{E^R_y} \nu_{\vec{w}} v^p \leq \int_{E^R_y} \left(\frac{M^{\otimes}(\vec{f})}{y} \right)^p \nu_{\vec{w}} \leq \frac{1}{y^p} \prod_{i=1}^{m} \left(\int_{E^R_y} (Mf_i)^{p_i} w_i \right)^{p/p_i}$$

$$\leq 2^{mp} y^{(m-1)p} \prod_{i=1}^{m} \left(\int_{\left\{ \frac{Mf_i}{v_i} > y \right\}} v_i^{p_i} w_i \right)^{p/p_i} \leq 2^{mp} C_1^p \dots C_m^p \frac{1}{y^p} \prod_{i=1}^{m} \|f_i\|^p_{L^{p_i,1}(w_i)} .$$

Iterating this result, we get that for each $y > 0$ and every natural number N,

$$\nu_{\vec{w}} v^p \left(\left\{ x \in \mathbb{R}^n : |x| < R,\ \frac{M^{\otimes}(\vec{f})(x)}{v(x)} > y \right\} \right) \leq 2^{mp} C_1^p \dots C_m^p \left(\sum_{j=0}^{N} \frac{1}{2^{jp}} \right) \frac{1}{y^p} \prod_{i=1}^{m} \|f_i\|^p_{L^{p_i,1}(w_i)}$$

$$+ \nu_{\vec{w}} v^p \left(\left\{ x \in \mathbb{R}^n : |x| < R,\ \frac{M^{\otimes}(\vec{f})(x)}{v(x)} > 2^{N+1} y \right\} \right),$$

and letting first N tend to infinity, and then R, the last term vanishes, and we conclude that

$$\lambda^{\nu_{\vec{w}} v^p}_{\frac{M^{\otimes}(\vec{f})}{v}}(y) \leq \frac{2^{(m+1)p}}{2^p - 1} C_1^p \dots C_m^p \frac{1}{y^p} \prod_{i=1}^{m} \|f_i\|^p_{L^{p_i,1}(w_i)} .$$

For $i = 1, \dots, m$, if we take $q_i > 1$ such that $w_i v^{p_i} \in A^{\mathcal{R}}_{q_i}$, in virtue of Lemma 3.1, we have that $w_i v_i^{p_i} \in A_{s_i}$, where $s_i > 1$ depends only on m, n, p_i, q_i, and $[w_i v_i^{p_i}]_{A_{s_i}} \lesssim_{m,n,p_i,q_i} [w_i v^{p_i}]^{2q_i}_{A^{\mathcal{R}}_{q_i}}$, so by Lemma 3.10, we have that $C_i \leq \mathscr{E}^n_{s_i,p_i}([w_i]_{A^{\mathcal{R}}_{p_i}}, C_{m,n,p_i,q_i}[w_i v^{p_i}]^{2q_i}_{A^{\mathcal{R}}_{q_i}})$, and hence, the desired result follows, with

$$C := \frac{2^{m+1}}{(2^p-1)^{1/p}} \prod_{i=1}^m \mathscr{E}^n_{s_i,p_i}([w_i]_{A^{\mathcal{R}}_{p_i}}, C_{m,n,p_i,q_i}[w_i v^{p_i}]^{2q_i}_{A^{\mathcal{R}}_{q_i}})),$$

which depends on the constants of the weights w_i and $w_i v^{p_i}$ in an increasing way. □

Remark 4.3. In the case when $p_1 = \cdots = p_m = 1$, the previous result is a corollary of Theorem 1.4 in [LOPi].

Observe that if we take weights $w_i \in A^{\mathcal{R}}_{p_i}$, $i = 1, \dots, m$, and $v \in RH_\infty$, then the hypotheses of Theorem 4.2 are satisfied.

The next result will be crucial to work with Calderón-Zygmund operators in the mixed restricted weak setting.

Theorem 4.4. *Let $0 < p < \infty$, let $\mathcal{S}$ be an η-sparse collection of cubes and let v, w be weights. Suppose that there exists $0 < \varepsilon \leq 1$ such that $\varepsilon < p$, $wv^{-\varepsilon} \in A_\infty$ and*

$$[v^{-\varepsilon}]_{RH_\infty(w)} := \sup_Q \frac{w(Q)}{wv^{-\varepsilon}(Q)} \|\chi_Q v^{-\varepsilon}\|_{L^\infty(w)} < \infty.$$

Then, there exists a constant C, independent of $\mathcal{S}$, such that the inequality

$$\left\| \frac{\mathcal{A}_{\mathcal{S}}(\vec{f})}{v} \right\|_{L^{p,\infty}(w)} \leq C \left\| \frac{\mathcal{M}(\vec{f})}{v} \right\|_{L^{p,\infty}(w)}$$

holds for every vector of measurable functions $\vec{f} = (f_1, \dots, f_m)$.

Proof. In virtue of Kolmogorov's inequalities, we obtain that

$$\left\| \frac{\mathcal{A}_{\mathcal{S}}(\vec{f})}{v} \right\|_{L^{p,\infty}(w)} \leq \sup_{0<w(F)<\infty} \left\| \frac{\mathcal{A}_{\mathcal{S}}(\vec{f})}{v} \chi_F \right\|_{L^{\varepsilon}(w)} w(F)^{\frac{1}{p}-\frac{1}{\varepsilon}},$$

where the supremum is taken over all measurable sets F with $0 < w(F) < \infty$. For one of such sets F, and $W := wv^{-\varepsilon}$, we have that

$$\begin{aligned}\left\| \frac{\mathcal{A}_{\mathcal{S}}(\vec{f})}{v} \chi_F \right\|^{\varepsilon}_{L^{\varepsilon}(w)} &\leq \int \sum_{Q \in \mathcal{S}} \chi_Q \prod_{i=1}^m \left(\frac{\fint_Q |f_i|}{v} \right)^{\varepsilon} \chi_F w \\ &= \sum_{Q\in\mathcal{S}} \prod_{i=1}^m \left(\fint_Q |f_i| \right)^{\varepsilon} \left(\frac{1}{W(3Q)} \int_Q \chi_F W \right) W(3Q) =: I.\end{aligned}$$

Since $W \in A_\infty$, there exists $r \geq 1$ such that $W \in A^{\mathcal{R}}_r$. Hence,

$$\sup_{E \subseteq Q} \frac{|E|}{|Q|} \left(\frac{W(Q)}{W(E)} \right)^{1/r} = \|W\|_{A^{\mathcal{R}}_r} < \infty,$$

where the supremum is taken over all cubes Q and all measurable sets $E \subseteq Q$. By hypothesis, $\mathcal{S}$ is η-sparse, so for each $Q \in \mathcal{S}$, $W(3Q) \leq (3^n \|W\|_{A^{\mathcal{R}}_r}/\eta)^r W(E_Q)$. Using this, we get that

$$\begin{aligned} I &\leq \left(\frac{3^n}{\eta} \|W\|_{A^{\mathcal{R}}_r} \right)^r \sum_{Q\in\mathcal{S}} \prod_{i=1}^m \left(\fint_Q |f_i| \right)^{\varepsilon} \left(\frac{1}{W(3Q)} \int_Q \chi_F W \right) W(E_Q) \\ &= \left(\frac{3^n}{\eta} \|W\|_{A^{\mathcal{R}}_r} \right)^r \sum_{Q\in\mathcal{S}} \int_{E_Q} \left(\prod_{i=1}^m \fint_Q |f_i| \right)^{\varepsilon} \left(\frac{1}{W(3Q)} \int_Q \chi_F W \right) W =: II. \end{aligned}$$

The sides of an n-dimensional cube have Lebesgue measure 0 in $\mathbb{R}^n$, so we can assume that the cubes in $\mathcal{S}$ are open. For $Q \in \mathcal{S}$ and $z \in E_Q$, we define $Q^z := Q(z, l_Q)$, the open cube of center z and side length twice the side length of Q. We have that $E_Q \subseteq Q \subseteq Q^z \subseteq 3Q$, so

$$\left(\prod_{i=1}^m \fint_Q |f_i| \right) \chi_{E_Q}(z) \leq \mathcal{M}(\vec{f})(z)$$

and

$$\frac{1}{W(3Q)}\int_Q \chi_F W \le \frac{1}{W(Q^z)}\int_{Q^z}\chi_F W \le M^c_W(\chi_F)(z).$$

Since the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint, and using Hölder's inequality with parameter $p/\varepsilon > 1$,

$$\begin{aligned} II &\le \left(\frac{3^n}{\eta}\|W\|_{A_r^{\mathcal{R}}}\right)^r \sum_{Q\in\mathcal{S}}\int_{E_Q}\mathcal{M}(\vec{f})^\varepsilon M^c_W(\chi_F)W \\ &\le \left(\frac{3^n}{\eta}\|W\|_{A_r^{\mathcal{R}}}\right)^r \left\|\left(\frac{\mathcal{M}(\vec{f})}{v}\right)^\varepsilon\right\|_{L^{p/\varepsilon,\infty}(w)} \|M^c_W(\chi_F)\|_{L^{(p/\varepsilon)',1}(w)} \\ &\le \frac{p}{p-\varepsilon}\left(\frac{3^n}{\eta}\|W\|_{A_r^{\mathcal{R}}}\right)^r \|M^c_W\|_{L^{(p/\varepsilon)',1}(w)}\, w(F)^{1-\frac{\varepsilon}{p}}\left\|\frac{\mathcal{M}(\vec{f})}{v}\right\|^\varepsilon_{L^{p,\infty}(w)}. \end{aligned}$$

Observe that for every measurable function g, $\|M^c_W(g)\|_{L^\infty(w)} \le \|g\|_{L^\infty(w)}$, and by standard arguments (see [G, Theorem 7.1.9]), it is easy to show that

$$\|M^c_W(g)\|_{L^{1,\infty}(w)} \le 24^n[v^{-\varepsilon}]_{RH_\infty(w)}\|g\|_{L^1(w)}.$$

In particular, and applying Marcinkiewicz interpolation theorem (see [BS, Theorem 4.13]), we conclude that

$$\|M^c_W\|_{L^{(p/\varepsilon)',1}(w)} \le c_{n,p,\varepsilon}[v^{-\varepsilon}]^{1-\frac{\varepsilon}{p}}_{RH_\infty(w)} < \infty.$$

Combining the previous estimates, we obtain that

$$\begin{aligned} &\left\|\frac{\mathcal{A}_\mathcal{S}(\vec{f})}{v}\chi_F\right\|_{L^\varepsilon(w)} w(F)^{\frac{1}{p}-\frac{1}{\varepsilon}} \\ &\le \left(\frac{p}{p-\varepsilon}\left(\frac{3^n}{\eta}\|W\|_{A_r^{\mathcal{R}}}\right)^r c_{n,p,\varepsilon}[v^{-\varepsilon}]^{1-\frac{\varepsilon}{p}}_{RH_\infty(w)}\right)^{1/\varepsilon}\left\|\frac{\mathcal{M}(\vec{f})}{v}\right\|_{L^{p,\infty}(w)}, \end{aligned}$$

and the desired result follows, with

$$C := \inf_{r\ge 1:\, W\in A_r^{\mathcal{R}}}\left(\frac{p}{p-\varepsilon}\left(\frac{3^n}{\eta}\|W\|_{A_r^{\mathcal{R}}}\right)^r c_{n,p,\varepsilon}[v^{-\varepsilon}]^{1-\frac{\varepsilon}{p}}_{RH_\infty(w)}\right)^{1/\varepsilon}.$$

□

Remark 4.5. For $0 < p \le 1$, if we take v such that $v^\delta \in A_\infty$ for some $\delta > 0$, and $w = uv^p$, with $u \in A_1$, the previous result can be established via an extrapolation type argument (see [OP, Theorem 1.1]).

Under the conditions that $0 < p \le 1$, and $w = uv^p$, we can find weights u and v that satisfy the hypotheses of Theorem 1.1 in [OP] but not the ones of Theorem 4.4, and vice versa. If we take a non-constant weight $u \in A_1$, and $v = u^{-1/p}$, then $v \in RH_\infty \subseteq A_\infty$, and $uv^p = 1$, but for every $0 < \varepsilon \le 1$ such that $\varepsilon < p$, we have that $v^{-\varepsilon} = u^{\varepsilon/p} \in A_1$, and since u is non-constant, $v^{-\varepsilon} \notin RH_\infty$. Similarly, if we take a non-constant weight $v \in A_1$, and $u = v^{-p}$, then $uv^p = 1$, and for every $\varepsilon > 0$, $uv^{p-\varepsilon} = v^{-\varepsilon} \in RH_\infty \subseteq A_\infty$, but $u \in RH_\infty$ and is non-constant, so $u \notin A_1$ (see Lemma 3.1).

The previous examples show that, sometimes, some of the hypotheses of Theorem 4.4 may be redundant. Let us be more precise on this fact. If $w \in A_\infty$, then $[v^{-\varepsilon}]_{RH_\infty(w)} < \infty$ implies that $wv^{-\varepsilon} \in A_\infty$. Indeed, given a cube $Q \subseteq \mathbb{R}^n$, and a measurable set $E \subseteq Q$, we have that

$$\frac{wv^{-\varepsilon}(E)}{wv^{-\varepsilon}(Q)} = \frac{1}{wv^{-\varepsilon}(Q)}\int_Q \chi_E wv^{-\varepsilon} \le \frac{w(E)}{wv^{-\varepsilon}(Q)}\|\chi_Q v^{-\varepsilon}\|_{L^\infty(w)} \le [v^{-\varepsilon}]_{RH_\infty(w)}\frac{w(E)}{w(Q)},$$

and since $w \in A_\infty$, there exist $\delta, C > 0$ such that

$$\frac{w(E)}{w(Q)} \le C\left(\frac{|E|}{|Q|}\right)^\delta,$$

so

$$\frac{wv^{-\varepsilon}(E)}{wv^{-\varepsilon}(Q)} \le C[v^{-\varepsilon}]_{RH_\infty(w)}\left(\frac{|E|}{|Q|}\right)^\delta,$$

and hence, $wv^{-\varepsilon} \in A_\infty$ (see [DMRO]).

The next application of Theorem 3.8 follows from the combination of Theorems 4.2 and 4.4, and gives us mixed restricted weak type bounds for multi-variable sparse operators that can also be deduced for other operators, such as multi-linear Calderón-Zygmund operators, using sparse domination techniques (see [L]).

Theorem 4.6. *Let $1 \leq p_1, \dots, p_m < \infty$, and let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let also $w_1, \dots, w_m$ be weights, with $w_i \in A_{p_i}^{\mathcal{R}}$, $i = 1, \dots, m$, and write $\nu_{\vec{w}} = w_1^{p/p_1} \dots w_m^{p/p_m}$. Let v be a weight such that $\nu_{\vec{w}} v^p$ is a weight, and $w_i v^{p_i} \in A_\infty$, $i = 1, \dots, m$. Suppose that there exists $0 < \varepsilon \leq 1$ such that $\varepsilon < p$, $\nu_{\vec{w}} v^{p-\varepsilon} \in A_\infty$ and $[v^{-\varepsilon}]_{RH_\infty(\nu_{\vec{w}} v^p)} < \infty$. Then, there exists a constant $C > 0$ such that the inequality*

$$\left\| \frac{T(\vec{f})}{v} \right\|_{L^{p,\infty}(\nu_{\vec{w}} v^p)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i,1}(w_i)}$$

holds for every vector of measurable functions $\vec{f} = (f_1, \dots, f_m)$, where T is either a sparse operator of the form

$$\mathcal{A}_{\mathcal{S}}(\vec{f}) := \sum_{Q \in \mathcal{S}} \prod_{i=1}^m \left(\fint_Q f_i \right) \chi_Q,$$

where $\mathcal{S}$ is an η-sparse collection of dyadic cubes, or any operator that can be conveniently dominated by such sparse operators, like m-linear ω-Calderón-Zygmund operators with ω satisfying the Dini condition.

Remark 4.7. In the case when $p_1 = \cdots = p_m = 1$, and T is a multi-linear Calderón-Zygmund operator, the previous result follows from Theorem 1.9 in [LOPi].

In general, there are examples of weights that satisfy the hypotheses of Theorem 4.6 apart from the constant weights. For instance, if $1 \leq p_1, \dots, p_m \leq m'$, we can take $w_i = (Mh_i)^{(1-p_i)/m}$, with $h_i \in L^1_{loc}(\mathbb{R}^n)$, $i = 1, \dots, m$, and $v = \nu_{\vec{w}}^{-1/p}$. Indeed, in virtue of Theorem 2.7 in [CGS], we have that $w_i \in A_{p_i}^{\mathcal{R}}$, $i = 1, \dots, m$, and $w_i v^{p_i} = \left(\prod_{j \neq i} (Mh_j)^{1/p_j'} \right)^{p_i/m} \in A_1$. Observe that $\nu_{\vec{w}} v^p = 1$, and $v = \left(\prod_{i=1}^m (Mh_i)^{1/p_i'} \right)^{1/m} \in A_1$, so for every $\varepsilon > 0$, $\nu_{\vec{w}} v^{p-\varepsilon} = v^{-\varepsilon} \in RH_\infty \subseteq A_\infty$.

The last application that we provide of Theorem 3.8 can be interpreted as a dual version of it, and generalizes [CS, Proposition 2.10].

Theorem 4.8. *Fix $p > 1$ and let u and v be weights such that $u \in A_p^{\mathcal{R}}$, $uv^p \in A_\infty$, and $[v^{-\varepsilon}]_{RH_\infty(uv^p)} < \infty$, for some $0 < \varepsilon \leq 1$. Then, there exists a constant C such that for every measurable function f,*

$$\left\| \frac{M(fuv^{p-1})}{u} \right\|_{L^{p',\infty}(u)} \leq C \|f\|_{L^{p',1}(uv^p)}.$$

Proof. It is known (see [L]) that there exist a collection $\{\mathscr{D}_\alpha\}_\alpha$ of 2^n general dyadic grids in $\mathbb{R}^n$, and a collection $\{\mathcal{S}_\alpha\}_\alpha$ of $\frac{1}{2}$-sparse families of cubes, with $\mathcal{S}_\alpha \subseteq \mathscr{D}_\alpha$, such that for every measurable function F,

$$MF \leq 2 \cdot 12^n \sum_{\alpha=1}^{2^n} \mathcal{A}_{\mathcal{S}_\alpha} |F|.$$

Hence,

$$\left\| \frac{M(fuv^{p-1})}{u} \right\|_{L^{p',\infty}(u)} \leq 2 \cdot 24^n \sum_{\alpha=1}^{2^n} \left\| \frac{\mathcal{A}_{\mathcal{S}_\alpha}(|f|uv^{p-1})}{u} \right\|_{L^{p',\infty}(u)}.$$

By duality, and self-adjointness of $\mathcal{A}_{\mathcal{S}_\alpha}$, and in virtue of Hölder's inequality, we have that

$$\begin{aligned} \left\| \frac{\mathcal{A}_{\mathcal{S}_\alpha}(|f|uv^{p-1})}{u} \right\|_{L^{p',\infty}(u)} &\leq p \sup_{\|g\|_{L^{p,1}(u)} \leq 1} \left\{ \int_{\mathbb{R}^n} \mathcal{A}_{\mathcal{S}_\alpha}(|f|uv^{p-1}) |g| \right\} \\ &= p \sup_{\|g\|_{L^{p,1}(u)} \leq 1} \left\{ \int_{\mathbb{R}^n} |f| uv^{p-1} \mathcal{A}_{\mathcal{S}_\alpha} |g| \right\} \\ &\leq p \sup_{\|g\|_{L^{p,1}(u)} \leq 1} \left\{ \left\| \frac{\mathcal{A}_{\mathcal{S}_\alpha} |g|}{v} \right\|_{L^{p,\infty}(uv^p)} \right\} \|f\|_{L^{p',1}(uv^p)}, \end{aligned}$$

and the desired result follows from Theorem 4.4 and Theorem 3.8. □

Remark 4.9. Observe that if $v = 1$, then in Theorem 4.8 we can take $\varepsilon = 1$, and $C = C_{n,p}[u]_{A_p^{\mathcal{R}}}^{p+1}$, and the dependence on u of the constant C is explicit, yielding a refined version of [CS, Proposition 2.10].

We would like to prove Theorem 4.8 without assuming that for some $0 < \varepsilon \leq 1$, $[v^{-\varepsilon}]_{RH_\infty(uv^p)} < \infty$. Unfortunately, at the time of writing, we don't know how to do it.

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