

AVERAGING OPERATORS ON DECREASING OR POSITIVE FUNCTIONS: EQUIVALENCE AND OPTIMAL BOUNDS

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ABSTRACT. We study the optimal bounds for the Hardy operator S minus the identity, as well as S and its dual operator S^* , on the full range $1 \leq p \leq \infty$, for the cases of decreasing, positive or general functions (in fact, these two kind of inequalities are equivalent for the appropriate cone of functions). For $1 < p \leq 2$, we prove that all these estimates are the same, but for $2 < p < \infty$, they exhibit a completely different behavior.

1. INTRODUCTION

Denote by $\mathcal{M}(\mathbb{R}^+)$ the class of all measurable functions on $\mathbb{R}^+ = (0, \infty)$, and for $f \in \mathcal{M}(\mathbb{R}^+)$, let us consider the classical Hardy averaging operator S and its dual S^* (see, e.g., [6]):

$$Sf(x) = \frac{1}{x} \int_0^x f(t) dt \quad \text{and} \quad S^*f(x) = \int_x^\infty \frac{f(t)}{t} dt.$$

In what follows, we will always assume that $S(|f|)(1) < \infty$ and $S^*(|f|)(1) < \infty$. These are natural conditions so that $S(f)$ and $S^*(f)$ are well defined. Moreover, such functions satisfy that $S^*(|f|) \in L^1(0, 1)$, which is an important property we will use later on. In fact,

$$(1) \quad \int_0^1 S^*(|f|)(t) dt = \int_0^1 |f(t)| dt + \int_1^\infty |f(t)| \frac{dt}{t}.$$

An important fact about S and S^* , which is a consequence of the classical Hardy inequalities, is that these operators are bounded in $L^p(\mathbb{R}^+)$

$$(2) \quad \|Sf\|_p \leq p' \|f\|_p, \quad \|S^*f\|_p \leq p \|f\|_p$$

and $p' = p/(p-1)$ and p are, respectively, the best constants. Furthermore, their norms are comparable (see [7, pp. 240, 244]),

$$(3) \quad \frac{1}{p'} \|Sf\|_p \leq \|S^*f\|_p \leq p \|Sf\|_p, \quad \text{for } 1 < p < \infty.$$

However, the optimal constants in (3) were actually found in [9] very recently:

Theorem 1.1. [9, Theorem 1.1] *Let $f \in \mathcal{M}(\mathbb{R}^+)$ be a positive function and assume $1 < p < \infty$. Then,*

$$(4) \quad (p-1) \|Sf\|_p \leq \|S^*f\|_p \leq (p-1)^{1/p} \|Sf\|_p,$$

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if $1 < p \leq 2$, and

$$(5) \quad (p-1)^{1/p} \|Sf\|_p \leq \|S^*f\|_p \leq (p-1) \|Sf\|_p,$$

if $2 \leq p < \infty$. Moreover, the constants in (4) and (5) are the best possible.

In recent years, extensions of Hardy's inequalities, for other operators, have been considered restricted to the cone of decreasing functions. In particular, sharp estimates for $S - \text{Id}$ and $f \downarrow$ were proved in [5], on the range $2 \leq p < \infty$ (see also [10] for a previous result when $p = 2, 3, 4, \dots$). The best constants in the full interval $1 < p < \infty$, again for decreasing functions, were finally characterized in [9]. One of the main techniques used in [9] is to show that the boundedness for $S - \text{Id}$ on decreasing functions is equivalent to estimates of the form $\|Sf\|_p \lesssim \|S^*f\|_p$, for positive functions $f \geq 0$. Inequalities of restricted type have been also obtained for all indexes $1 < p < \infty$ [11].

To complete this framework, it is interesting to also recall that, motivated by the well-known Iwaniec's conjecture [8] regarding the norm of the Beurling operator B on $L^p(\mathbb{C})$ (see [1, 4] for more information), and using the fact that, for radial functions $F(z) = f(|z|^2)$, the following equality holds

$$BF(z) = \frac{\bar{z}}{z} (S - \text{Id})f(|z|^2), \quad z \in \mathbb{C},$$

then the norm of B on such radial functions is equivalent to the norm of $S - \text{Id}$ for general functions f . In this case, optimal constants are only known if $1 < p \leq 2$ (see [2]).

In this work, we will describe the best constants in all known cases, for the inequalities involving $S - \text{Id}$, S and S^* , in all three different cones of functions (general f , positive $f \geq 0$ or decreasing $f \downarrow$). For the estimates still open, we will give the best bounds available.

The following result, based on the arguments given in [9] for decreasing and positive functions, is an important tool we will use to prove the equivalence of the optimal constants for general functions. In what follows, we will assume X and Y to be lattice quasi-Banach spaces over \mathbb{R}^+ (e.g., L^p , $1 \leq p \leq \infty$ or $L^{1,\infty}$).

Lemma 1.2. *Let X and Y be two lattice quasi-Banach spaces over \mathbb{R}^+ . If for every function f , the following inequality holds*

$$\|Sf - f\|_Y \leq C_{X,Y} \|f\|_X,$$

then

$$\|Sf\|_Y \leq C_{X,Y} \|S^*f\|_X.$$

Proof. Let f be a general function and let $g(x) = S^*f(x)$. Then, as a consequence of (1), it is easy to see that we can use Fubini's theorem, and hence

$$Sg(x) - g(x) = S(S^*f)(x) - S^*f(x) = Sf(x) + S^*f(x) - S^*f(x) = Sf(x).$$

Thus, by hypothesis,

$$\|Sf\|_Y = \|Sg - g\|_Y \leq C_{X,Y} \|g\|_X = C_{X,Y} \|S^*f\|_X. \quad \square$$

The following table summarizes our main goals. For a given norm (or quasi-norm) $\|\cdot\|$ we are going to describe the best known constants for each inequality on the first column, and for all three different cones of the first row.

Norm: $\ \cdot\ $	$f \downarrow$	$f \geq 0$	f
$\ Sf - f\ \lesssim \ f\ $	A	B	C
$\ Sf\ \lesssim \ S^*f\ $	D	E	F

TABLE 1. Best constants for different cones of functions.

Observe that we easily get the inequalities $A \leq B \leq C$; $D \leq E \leq F$, and, using Lemma 1.2, we also have that $F \leq C$. Moreover, in all cases we are going to also obtain that $A = E$.

In Section 2 we review the optimal constants on Table 1, for the range $1 < p < 2$ and observe that they are all equal to the value $1/(p - 1)$. Section 3 deals with the range $2 \leq p < \infty$. Here we show that the constants behave more wildly (they are all different, except for the trivial cases). Our main results are the following: In Theorem 3.2 we find the sharp estimate for $S - \text{Id}$, on the cone of positive functions $f \geq 0$ and in Theorem 3.3 we also obtain the best bound relating S and S^* , for f decreasing, whenever $p = 2, 3, 4, \dots$. In Section 4, we study what happens at the endpoints $p = 1$ and $p = \infty$. In Section 5 we make some final remarks about the constants C , D and F , which are still unknown in its full generality, for the range $2 < p < \infty$.

2. SHARP ESTIMATES FOR $1 < p < 2$

Sharp estimates for the norm of the Hardy operator minus the identity in the cone of decreasing functions, on the range $1 < p < 2$, were proved in [9]. The main techniques used in that paper consist of establishing the equivalence of optimal constants between L^p norms of the functions Sf and S^*f , $f \geq 0$, and L^p norms of the functions $S\varphi - \varphi$ and φ , where φ is a decreasing function on \mathbb{R}^+ . The result is the following:

Theorem 2.1. [9, Theorems 1.1 and 1.2] *Let $1 < p < 2$.*

(i) *If $f \geq 0$ on \mathbb{R}^+ , then the inequality*

$$(6) \quad \|Sf\|_p \leq \frac{1}{p-1} \|S^*f\|_p,$$

is sharp.

(ii) *If f is decreasing on \mathbb{R}^+ , then the inequality*

$$(7) \quad \|Sf - f\|_p \leq \frac{1}{p-1} \|f\|_p$$

is sharp.

Remark 2.2. The optimality of (6) was proved by considering the family of decreasing functions $g_\varepsilon(x) = x^{\varepsilon-1/p} \chi_{(0,1)}(x)$, for $0 < \varepsilon \leq 1/p$. Therefore, we also obtain

that the best constant in (6), for decreasing functions, is equal to $1/(p-1)$. Thus, in the notation of Table 1, we have that $A = D = E = 1/(p-1)$.

Estimate (7), for general functions $f \in L^p(\mathbb{R}^+)$, and $1 < p < 2$, was established in [2]. The proof makes use of very subtle arguments and is more involved than the case of decreasing functions.

Theorem 2.3. [2, Theorem 4.1] *If $1 < p < 2$, and $f \in L^p(\mathbb{R}^+)$, then the inequality*

$$(8) \quad \|Sf - f\|_p \leq \frac{1}{p-1} \|f\|_p$$

is sharp.

Remark 2.4. We observe that using (7) and (8), we have that $\|Sf - f\|_p \leq \frac{1}{p-1} \|f\|_p$ is also optimal for $f \geq 0$. Thus, in the notation of Table 1, we also have that $A = B = C = D = E = 1/(p-1)$.

Corollary 2.5. *Let $1 < p < 2$, then, for any $f \in L^p(\mathbb{R}^+)$ the following inequality is optimal,*

$$\|Sf\|_p \leq \frac{1}{p-1} \|S^*f\|_p.$$

Proof. It is a direct consequence of Theorems 2.1 and 2.3, and Lemma 1.2. \square

Thus, we have finally proved that on the range $1 < p < 2$ all constants are the same: $A = B = C = D = E = F = 1/(p-1)$.

3. SHARP ESTIMATES FOR $2 \leq p < \infty$

In [5], as a particular case of the sharp constants obtained for the norm of the operator $S - \text{Id}$, acting on the cone of decreasing functions in weighted L^p spaces, we were able to prove that for any $p \geq 2$ and $f \downarrow$, then

$$\|Sf - f\|_p \leq (p-1)^{-1/p} \|f\|_p,$$

and the constant $A = (p-1)^{-1/p}$ is optimal. For $p \geq 2$ a natural number, this was previously obtained in [10]. Again, using (5), it also holds that $E = (p-1)^{-1/p}$; i.e., for any positive function $f \geq 0$

$$(9) \quad \|Sf\|_p \leq (p-1)^{-1/p} \|S^*f\|_p.$$

For the proof of our main result on the norm of $S - \text{Id}$ acting on the cone of positive functions in $L^p(\mathbb{R}^+)$, the following lemma will be useful.

Lemma 3.1. *If $p \geq 2$ and $0 < t \leq s$, then*

(i)

$$f_1(s, t) = s^p \left[\left(1 - \frac{t}{s}\right)^p - 1 + \frac{pt}{s} \right] - t^p \geq 0$$

and

(ii)

$$f_2(s, t) = pt^{p-1}(t-s) - t^p + s^p \left[1 - \left(1 - \frac{t}{s}\right)^p \right] \geq 0.$$

Proof. For the proof of part (i), by homogeneity, it is enough to consider f_1 as a function of the variable $0 \leq r = t/s \leq 1$ and define

$$g_1(r) = f_1(1, r) = (1 - r)^p - 1 + pr - r^p.$$

Then, $g_1(0) = 0$ and g_1 is increasing in $[0, 1]$, since its derivative

$$g_1'(r) = p(1 - r^{p-1} - (1 - r)^{p-1}) \geq 0,$$

because $p \geq 2$ and $0 \leq r \leq 1$.

Similarly, to prove (ii), let us write f_2 in terms of the variable $0 \leq r = t/s \leq 1$ and define

$$g_2(r) = f_2(1, r) = (p - 1)r^p - pr^{p-1} - (1 - r)^p + 1.$$

In this case, $g_2(0) = g_2(1) = 0$ and its derivative

$$g_2'(r) = p(p - 1)r^{p-2}(r - 1) + p(1 - r)^{p-1}$$

has a unique zero at the point $r_0 = 1/(1 + (p - 1)^{1/(p-2)})$, which is a local maximum, since $p \geq 2$. As a consequence, g_2 is increasing in $[0, r_0]$ and decreasing in $(r_0, 1]$, so $g_2(r) \geq 0$, for $r \in [0, 1]$. \square

It is well-known that the operator $S - \text{Id}$ is an isometry in $L^2(\mathbb{R}^+)$ (see [6]). Hence, $A = B = C = 1$, if $p = 2$. Concerning $p > 2$, we have the following result:

Theorem 3.2. *If $2 < p < \infty$, and f is a positive function in $L^p(\mathbb{R}^+)$, then the following sharp estimate holds*

$$\|Sf - f\|_p \leq \|f\|_p.$$

Proof. By a density argument, we can restrict ourselves to the case of f a positive function of compact support that agrees with a polynomial in its support. Then, $Sf - f$ is also a continuous function on \mathbb{R}^+ , except possibly at one point where it would have a jump discontinuity, has a finite number of changes of sign, and no zeros outside the support of f .

Then, let us consider $I_j = (a_{j-1}, a_j)$, $1 \leq j \leq N$ be a finite partition of \mathbb{R}^+ with $a_0 = 0$, $a_N = \infty$, in such a way that the sign of $Sf - f$ is constant in the interval I_j and it changes between two consecutive intervals. Let J be the collection of indexes $1 \leq j \leq N$ such that $Sf - f > 0$ on I_j , and $J' = \{1, \dots, N\} \setminus J$; that is, $Sf - f < 0$ on I_j , $j \in J'$.

For $\alpha \in \mathbb{R}$, we will use the following representation in power series of the function $(1 + t)^\alpha$:

$$(10) \quad (1 + t)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} t^k, \quad |t| < 1,$$

where, for every positive integer k ,

$$\binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - k)}{k!},$$

and $\binom{\alpha}{0} = 1$.

In every interval $I_j = (a_{j-1}, a_j)$, with $j \in J$, since $-1 < (f - Sf)/Sf < 0$, using the binomial identity (10), we have

$$\begin{aligned} \int_{I_j} f^p(x) dx &= \int_{I_j} \left(\frac{f(x) - Sf(x)}{Sf(x)} + 1 \right)^p (Sf(x))^p dx \\ &= \int_{I_j} \sum_{k=0}^{\infty} \binom{p}{k} \left(\frac{f(x) - Sf(x)}{Sf(x)} \right)^k (Sf(x))^p dx. \end{aligned}$$

Then, if we denote by $u(t) = Sf(t)$, we have that $f(t) = (tu(t))' = tu'(t) + u(t)$, and hence this last integral is equal to

$$\begin{aligned} \int_{I_j} \sum_{k=0}^{\infty} \binom{p}{k} \left(\frac{tu'(t)}{u(t)} \right)^k (u(t))^p dt &= \int_{I_j} (u^p(t) + ptu^{p-1}(t)u'(t)) dt \\ &\quad + \int_{I_j} \sum_{k=2}^{\infty} \binom{p}{k} \left(\frac{tu'(t)}{u(t)} \right)^k (u(t))^p dt. \end{aligned}$$

Integrating the first term of the right-hand side, we obtain

$$(11) \quad \int_{I_j} (u^p(t) + ptu^{p-1}(t)u'(t)) dt = a_j u^p(a_j) - a_{j-1} u^p(a_{j-1}),$$

where we understand that if $j = N$, and hence $a_N = \infty$, then $a_N u^p(a_N) = \lim_{t \rightarrow \infty} tu^p(t) = 0$. Therefore, with the original notation, we have obtained that, for every interval $I_j = (a_{j-1}, a_j)$, such that $j \in J$,

$$\begin{aligned} \int_{I_j} (f^p(x) - (Sf(x) - f(x))^p) dx &= a_j (Sf(a_j))^p - a_{j-1} (Sf(a_{j-1}))^p \\ (12) \quad &\quad + \int_{I_j} \sum_{k=2}^{\infty} \binom{p}{k} \left(\frac{f(x) - Sf(x)}{Sf(x)} \right)^k (Sf(x))^p dx \\ &\quad - \int_{I_j} (Sf(x) - f(x))^p dx. \end{aligned}$$

Similarly, in every interval $I_j = (a_{j-1}, a_j)$, with $j \in \tilde{J}$, since $-1 < -Sf/f < 0$, using again the binomial identity (10), we obtain

$$\begin{aligned} \int_{I_j} (f(x) - Sf(x))^p dx &= \int_{I_j} \left(\frac{-Sf(x)}{f(x)} + 1 \right)^p (f(x))^p dx \\ &= \int_{I_j} \sum_{k=0}^{\infty} \binom{p}{k} \left(\frac{-Sf(x)}{f(x)} \right)^k (f(x))^p dx. \end{aligned}$$

Therefore, using (11) expressed in terms of $u(t) = Sf(t)$ and $f(t) = (tu(t))' = tu'(t) + u(t)$, we obtain that for every interval $I_j = (a_{j-1}, a_j)$, such that $j \in \tilde{J}$,

$$(13) \quad \int_{I_j} (f^p(x) - (f(x) - Sf(x))^p) dx = a_j(Sf(a_j))^p - a_{j-1}(Sf(a_{j-1}))^p \\ + \int_{I_j} (p-1)(Sf(x))^p - pf(x)(Sf(x))^{p-1} dx \\ - \int_{I_j} \sum_{k=1}^{\infty} \binom{p}{k} \left(\frac{-Sf(x)}{f(x)} \right)^k (f(x))^p dx.$$

Summing up the equalities proved in (12) for every $j \in J$, and in (13) for every $j \in \tilde{J}$,

$$(14) \quad \int_0^{\infty} (f^p(x) - |f(x) - Sf(x)|^p) dx = \sum_{j=1}^N (a_j(Sf(a_j))^p - a_{j-1}(Sf(a_{j-1}))^p) \\ + \sum_{j \in J} \int_{I_j} \left[\sum_{k=2}^{\infty} \binom{p}{k} \left(\frac{f(x) - Sf(x)}{Sf(x)} \right)^k (Sf(x))^p - (Sf(x) - f(x))^p dx \right] \\ + \sum_{j \in \tilde{J}} \int_{I_j} \left[(p-1)(Sf(x))^p - pf(x)(Sf(x))^{p-1} - \sum_{k=1}^{\infty} \binom{p}{k} \left(\frac{-Sf(x)}{f(x)} \right)^k (f(x))^p dx \right].$$

In order to see that this last expression is positive, we look at the three terms of the right-hand side that can be simplified as follows. The first one,

$$\sum_{j=1}^N (a_j(Sf(a_j))^p - a_{j-1}(Sf(a_{j-1}))^p) = \lim_{r \rightarrow \infty} rSf(r)^p - \lim_{r \rightarrow 0^+} rSf(r)^p = 0,$$

due to the fact that f has compact support. For the second one, we observe that, since $j \in J$,

$$\sum_{k=2}^{\infty} \binom{p}{k} \left(\frac{f(x) - Sf(x)}{Sf(x)} \right)^k (Sf(x))^p - (Sf(x) - f(x))^p \\ = (Sf(x))^p \left[\left(1 - \frac{Sf(x) - f(x)}{Sf(x)} \right)^p - 1 + p \frac{Sf(x) - f(x)}{Sf(x)} \right] - (Sf(x) - f(x))^p$$

and this last function is positive, due to Lemma 3.1 (i) for $s = Sf(x) > 0$ and $t = Sf(x) - f(x) > 0$, where $0 < t < s$.

Finally the third one, since $j \in \tilde{J}$, then

$$(p-1)(Sf(x))^p - pf(x)(Sf(x))^{p-1} - \sum_{k=1}^{\infty} \binom{p}{k} \left(\frac{-Sf(x)}{f(x)} \right)^k (f(x))^p \\ = p(Sf(x))^{p-1}(Sf(x) - f(x)) - (Sf(x))^p + (f(x))^p \left[1 - \left(1 - \frac{Sf(x)}{f(x)} \right)^p \right]$$

and this last function is positive, due to Lemma 3.1 (ii) for $s = f(x) > 0$ and $t = Sf(x) > 0$, since $0 < t < s$. These observations allow us to conclude from (14)

that

$$(15) \quad \|Sf - f\|_p \leq \|f\|_p.$$

Finally, in order to check that this last estimate is sharp, let us take $f_r(x) = \chi_{[1, r+1)}(x)$, with $r > 0$. For this function f_r ,

$$(Sf_r - f_r)(x) = -\frac{1}{x}\chi_{[1, 1+r)}(x) + \frac{r}{x}\chi_{(r+1, \infty)}(x).$$

Thus

$$\lim_{r \rightarrow 0^+} \frac{\|Sf_r - f_r\|_p^p}{\|f_r\|_p^p} = \lim_{r \rightarrow 0^+} \frac{1}{p-1} \frac{(r+1)^{p-1} - 1 + r^p}{r(r+1)^{p-1}} = 1.$$

This shows that the constant 1 in the right-hand side of (15) is optimal. \square

Equivalently, Theorem 3.2 proves that, for every $p \geq 2$, we have that $B = 1$. Continuing with the study of best estimates, we observe that, for $p = 2$, and any general function f , using the equalities $SS^* = S^*S = S + S^*$, we can prove

$$\|Sf\|_2 = \|S^*f\|_2.$$

In particular, this holds for f decreasing. Now, for other values of $p \geq 2$ and f decreasing, we have the following theorem concerning the constant D .

Theorem 3.3. *Let $p \geq 2$ be a natural number. Then, for any decreasing function $f \in L^p(\mathbb{R}^+)$, the following inequality is sharp*

$$\|Sf\|_p \leq \left(\frac{p'}{p!}\right)^{1/p} \|S^*f\|_p.$$

Proof. By using the density of simple and decreasing functions in the corresponding cone in $L^p(\mathbb{R}^+)$, we can restrict ourselves to prove the inequality for functions

$$f_N(x) = \sum_{k=1}^N b_k \chi_{(0, a_k)}(x),$$

with $b_j \geq 0$, $j = 1, \dots, N$, and $0 \leq a_1 \leq a_2 \leq \dots \leq a_N$. Easy calculations show that

$$Sf_N(x) = \sum_{j=1}^N b_j \left(\chi_{(0, a_j)}(x) + \frac{a_j}{x} \chi_{(a_j, \infty)}(x) \right)$$

and

$$S^*f_N(x) = \sum_{j=1}^N b_j \log\left(\frac{a_j}{x}\right) \chi_{(0, a_j)}(x).$$

Thus, our goal is to check the following sharp inequality, for every function f_N

$$(16) \quad \int_0^\infty (Sf_N)^p(x) dx \leq \frac{p'}{p!} \int_0^\infty (S^*f_N)^p(x) dx.$$

Looking at both sides of (16) as a homogeneous polynomial of degree N in the variables b_1, \dots, b_N , we observe that it is enough to prove the estimate for every term $\prod_{j=1}^N b_j^{\alpha_j}$ associated to the multi-index $\bar{\alpha} = (\alpha_j)_{j=1}^N$, with $0 \leq \alpha_j \leq p$ and

$\sum_{i=1}^p \alpha_i = p$. In order to do that, notice that for a fixed multi-index $\bar{\alpha} = (\alpha_j)_{j=1}^N$, the terms on the left-hand side of (16) correspond to the following integral

$$(17) \quad \int_0^\infty \prod_{j=1}^N \left(\chi_{(0, a_j)}(x) + \left(\frac{a_j}{x} \right)^{\alpha_j} \chi_{(a_j, \infty)} \right) dx,$$

and, similarly, the right-hand side for the corresponding term of the same multi-index is equal to

$$\int_0^\infty \prod_{j=1}^N \log^{\alpha_j} \left(\frac{a_j}{x} \right) \chi_{(0, a_j)}^{\alpha_j}(x) dx = \int_0^{a_{j_0}} \prod_{j=j_0}^N \log^{\alpha_j} \left(\frac{a_j}{x} \right) dx,$$

where $j_0 = \min\{j : \alpha_j \neq 0\}$.

To find an explicit expression of the integral in (17), we first introduce the following notation. For $\bar{\alpha} = (\alpha_1, \dots, \alpha_N)$ a fixed multi-index, with $0 \leq \alpha_j \leq p$ and $\sum_{i=1}^p \alpha_i = p$, let us define $A_j = \sum_{i=1}^j \alpha_i$, for any $1 \leq j \leq N$. Then, it is clear that $0 \leq A_1 \leq \dots \leq A_N = p$, and (17) is equal to

$$(18) \quad a_1 + \sum_{j=1}^N \int_{a_j}^{a_{j+1}} \frac{\prod_{k=1}^j a_k^{\alpha_k}}{x^{A_j}} dx,$$

where $a_0 = 0$ and $a_{N+1} = \infty$. As before, let j_0 be the smallest index j such that $\alpha_j \neq 0$. We will distinguish the following two cases:

If $\alpha_{j_0} = 1$, let us define j_1 as the first index such that $A_{j_1} \geq 2$. Hence, $A_k = 1$, for all $j_0 \leq k \leq j_1 - 1$ and $A_k \geq 2$, for $j_1 \leq k \leq N$. Note that for any $j < j_0$, since $\alpha_j = 0$, then $a_j^{\alpha_j} = 1$. Therefore, (18) is equal to

$$a_{j_0} + a_{j_0} \sum_{j=j_0}^{j_1-1} \log \left(\frac{a_{j+1}}{a_j} \right) + \sum_{j=j_1}^N \left(\prod_{k=j_0}^j a_k^{\alpha_k} \right) \frac{a_j^{1-A_j} - a_{j+1}^{1-A_j}}{A_j - 1},$$

taking into account that for $j = N$, we have that $A_N = p > 1$ and $a_{N+1}^{1-A_N} = 0$.

On the other hand, if $\alpha_{j_0} \geq 2$, there are no terms with $A_j = 1$ and the integral in (18) is equal to

$$a_{j_0} + \sum_{j=j_0}^N \left(\prod_{k=j_0}^j a_k^{\alpha_k} \right) \frac{a_j^{1-A_j} - a_{j+1}^{1-A_j}}{A_j - 1}.$$

As a consequence, in order to obtain the estimate (16), we will prove that for every multi-index $\bar{\alpha}$, with $\alpha_{j_0} = 1$, the following estimate is sharp

$$(19) \quad \begin{aligned} & a_{j_0} + a_{j_0} \sum_{j=j_0}^{j_1-1} \log \left(\frac{a_{j+1}}{a_j} \right) + \sum_{j=j_1}^N \left(\prod_{k=j_0}^j a_k^{\alpha_k} \right) \frac{a_j^{1-A_j} - a_{j+1}^{1-A_j}}{A_j - 1} \\ & \leq \frac{p'}{p!} \int_0^{a_{j_0}} \prod_{j=j_0}^N \log^{\alpha_j} \left(\frac{a_j}{x} \right) dx = \frac{p'}{p!} a_{j_0} \int_0^1 \prod_{j=j_0}^N \log^{\alpha_j} \left(\frac{a_j}{a_{j_0} x} \right) dx. \end{aligned}$$

A second kind of bounds will consist in proving that the following inequality is optimal and holds for every multi-index $\bar{\alpha}$, with $\alpha_{j_0} \geq 2$,

$$(20) \quad a_{j_0} + \sum_{j=j_0}^N \left(\prod_{k=j_0}^j a_k^{\alpha_k} \right) \frac{a_j^{1-A_j} - a_{j+1}^{1-A_j}}{A_j - 1} \leq \frac{p'}{p!} a_{j_0} \int_0^1 \prod_{j=j_0}^N \log^{\alpha_j} \left(\frac{a_j}{a_{j_0} x} \right) dx.$$

First of all, let us prove (20). Since in the expression to be estimated, all the variables a_j involved correspond to $j_0 \leq j \leq N$, we can assume, without loss of generality, that $j_0 = 1$. We adopt the notation $\alpha_1 = \alpha$ and separate in (20) the term with $j = j_0 = 1$. Let us consider, then, the following function Φ of N variables (a_1, \dots, a_N) restricted to the set $0 < a_1 \leq \dots \leq a_N$,

$$\begin{aligned} \Phi(a_1, \dots, a_N) &= a_1 + \frac{a_1^\alpha}{\alpha - 1} (a_1^{1-\alpha} - a_2^{1-\alpha}) \\ &\quad + \sum_{j=2}^N \left(\prod_{k=2}^j a_k^{\alpha_k} \right) \frac{a_j^{1-A_j} - a_{j+1}^{1-A_j}}{A_j - 1} - \frac{p'}{p!} a_1 \int_0^1 \log^\alpha \left(\frac{1}{x} \right) \prod_{j=2}^N \log^{\alpha_j} \left(\frac{a_j}{a_1 x} \right) dx. \end{aligned}$$

With this notation, (20) is equivalent to proving that $\Phi(a_1, \dots, a_N) \leq 0$, on the set $0 < a_1 \leq \dots \leq a_N$.

Dividing all the expression above by $a_1 > 0$, and renaming the variables as $r_j = a_1/a_j$, $1 \leq j \leq N$, our problem is equivalent to consider the function

$$\begin{aligned} \Psi(1, r_2, \dots, r_N) &= \frac{\alpha}{\alpha - 1} - \frac{r_2^{\alpha-1}}{\alpha - 1} \\ &\quad + \sum_{j=2}^N \left(\prod_{k=2}^j r_k^{-\alpha_k} \right) \frac{r_j^{A_j-1} - r_{j+1}^{A_j-1}}{A_j - 1} - \frac{p'}{p!} \int_0^1 \log^\alpha \left(\frac{1}{x} \right) \prod_{j=2}^N \log^{\alpha_j} \left(\frac{1}{r_j x} \right) dx, \end{aligned}$$

restricted to the set S where $0 < r_N \leq \dots \leq r_2 \leq 1 = r_1$ and with the assumption $r_{N+1} = 0$. Also, without loss of generality, we assume $\alpha_N \geq 1$, otherwise there is no dependence of the variable r_N . We will prove that on S the function attains its maximum at the point $1 = r_2 = \dots = r_N$. Since

$$\Psi(1, \dots, 1) = 1 + \frac{1}{p-1} - \frac{p'}{p!} \int_0^1 \log^p \left(\frac{1}{x} \right) dx = p' - \frac{p'}{p!} \Gamma(p+1) = 0,$$

this would prove that the inequality (20) is sharp. We are going to show that Ψ is an increasing function with respect to the variable r_N on the set S . Computing its partial derivative, we obtain

$$\begin{aligned} \frac{\partial \Psi}{\partial r_N}(1, r_2, \dots, r_n) &= -\frac{\alpha_N}{p-1} \prod_{k=2}^{N-1} r_k^{-\alpha_k} r_N^{A_{N-1}-2} \\ &\quad + \frac{p'}{p!} \frac{\alpha_N}{r_N} \int_0^1 \log^\alpha \left(\frac{1}{x} \right) \prod_{j=2}^{N-1} \log^{\alpha_j} \left(\frac{1}{r_j x} \right) \log^{\alpha_N-1} \left(\frac{1}{r_N x} \right) dx. \end{aligned}$$

Therefore, in order to prove that this derivative is positive on the set S , we must justify that the following inequality holds on S ,

$$\prod_{k=2}^{N-1} r_k^{-\alpha_k} r_N^{A_{N-1}-1} \leq \frac{1}{(p-1)!} \int_0^1 \log^\alpha \left(\frac{1}{x} \right) \prod_{j=2}^{N-1} \log^{\alpha_j} \left(\frac{1}{r_j x} \right) \log^{\alpha_N-1} \left(\frac{1}{r_N x} \right) dx.$$

But, since $r_N \leq r_k \leq 1$, for $k \geq N - 1$,

$$\begin{aligned} \prod_{k=2}^{N-1} r_k^{-\alpha_k} r_N^{A_{N-1}-1} &= \prod_{k=2}^{N-1} \left(\frac{r_N}{r_k} \right)^{\alpha_k} \leq 1 \\ &= \frac{1}{(p-1)!} \int_0^1 \log^\alpha \left(\frac{1}{x} \right) \prod_{j=2}^{N-1} \log^{\alpha_j} \left(\frac{1}{x} \right) \log^{\alpha_{N-1}} \left(\frac{1}{x} \right) dx \\ &\leq \frac{1}{(p-1)!} \int_0^1 \log^\alpha \left(\frac{1}{x} \right) \prod_{j=2}^{N-1} \log^{\alpha_j} \left(\frac{1}{r_j x} \right) \log^{\alpha_{N-1}} \left(\frac{1}{r_N x} \right) dx, \end{aligned}$$

and we are done. In this way, we will put $r_N = r_{N-1}$ and consider the function of $N - 2$ variables $\Psi(1, r_2, \dots, r_{N-1}, r_{N-1})$ on the set $0 < r_{N-1} \leq \dots \leq r_2$, which will also define an increasing function on this set with respect to the variable r_{N-1} , and then we proceed recursively by considering, for any $2 \leq k \leq N - 1$, the function of $k - 1$ variables

$$\begin{aligned} \Psi(1, r_2, \dots, r_k, \dots, r_k) &= \frac{\alpha}{\alpha - 1} - \frac{r_2^{\alpha-1}}{\alpha - 1} + \sum_{j=2}^{k-1} \left(\prod_{i=2}^j r_i^{-\alpha_i} \right) \frac{r_j^{A_j-1} - r_{j+1}^{A_j-1}}{A_j - 1} \\ &+ \frac{\prod_{i=2}^{k-1} r_i^{-\alpha_i} \prod_{i=k}^N r_k^{-\alpha_i}}{A_N - 1} r_k^{A_N-1} - \frac{p'}{p!} \int_0^1 \log^\alpha \left(\frac{1}{x} \right) \prod_{j=2}^{k-1} \log^{\alpha_j} \left(\frac{1}{r_j x} \right) \log^{p-A_{k-1}} \left(\frac{1}{r_k x} \right) dx, \end{aligned}$$

and its partial derivative with respect to the last variable r_k is equal to

$$\begin{aligned} &- \frac{A_{k-1} - p}{p - 1} \prod_{i=2}^{k-1} r_i^{-\alpha_i} r_k^{A_{k-1}-2} \\ &+ \frac{p' p - A_{k-1}}{p! r_k} \int_0^1 \log^\alpha \left(\frac{1}{x} \right) \prod_{j=2}^{k-1} \log^{\alpha_j} \left(\frac{1}{r_j x} \right) \log^{p-A_{k-1}-1} \left(\frac{1}{r_k x} \right) dx. \end{aligned}$$

Similarly, as in the previous case, we have that this derivative is positive on the set $0 < r_k \leq \dots \leq r_2 \leq 1$ due to the chain of inequalities

$$\begin{aligned} \prod_{i=2}^{k-1} r_i^{-\alpha_i} r_k^{A_{k-1}-1} &= \prod_{k=2}^{k-1} \left(\frac{r_k}{r_i} \right)^{\alpha_i} \leq 1 = \frac{1}{(p-1)!} \int_0^1 \log^\alpha \left(\frac{1}{x} \right) \log^{p-\alpha-1} \left(\frac{1}{x} \right) dx \\ &\leq \frac{1}{(p-1)!} \int_0^1 \log^\alpha \left(\frac{1}{x} \right) \prod_{j=2}^{k-1} \log^{\alpha_j} \left(\frac{1}{r_j x} \right) \log^{p-A_{k-1}-1} \left(\frac{1}{r_k x} \right) dx. \end{aligned}$$

By this procedure, we can finally reduce the number of the variables until we just consider, on the set $0 < r_2 \leq 1$, the function

$$\Psi(1, r_2, \dots, r_2) = \frac{\alpha - r_2^{\alpha-1}}{\alpha - 1} + \frac{\prod_{i=2}^N r_2^{-\alpha_i} r_2^{p-1}}{p - 1} - \frac{p'}{p!} \int_0^1 \log^\alpha \left(\frac{1}{x} \right) \prod_{j=2}^N \log^{\alpha_j} \left(\frac{1}{r_2 x} \right) dx.$$

As before, we can prove that its derivative in $r_2 \in (0, 1]$ is also positive and, therefore, on the set $0 < r_N \leq \dots \leq r_2 \leq 1$

$$\Psi(1, r_2, \dots, r_N) \leq \Psi(1, \dots, 1) = 0,$$

which ends the proof of (20).

We now proceed with estimate (19). As in the previous case, we can assume that $j_0 = 1$, since all the variables a_j involved in (19) are those with $j_0 \leq j \leq N$. Therefore, we are considering a multi-index $\bar{\alpha}$, with $\alpha_1 = 1$, $A_j = 1$, $1 \leq j \leq j_1 - 1$, $A_j \geq 2$, for $j_1 \leq j \leq N$. Then, dividing the inequality by a_1 and defining the variables $r_i = a_1/a_i$, $1 \leq i \leq N$, we have to prove the following inequality, provided that $0 < r_N \leq \dots \leq r_2 \leq r_1 = 1$:

$$1 + \log\left(\frac{1}{r_{j_1}}\right) + \sum_{j=j_1}^N \left(\prod_{k=j_1}^j r_k^{-\alpha_k} \right) \frac{r_j^{A_j-1} - r_{j+1}^{A_j-1}}{A_j - 1} \leq \frac{p'}{p!} \int_0^1 \log\left(\frac{1}{x}\right) \prod_{j=j_1}^N \log^{\alpha_j}\left(\frac{1}{r_j x}\right) dx.$$

Note that the assumptions on the multi-index $\bar{\alpha}$ imply that just the variables r_j , with $j_1 \leq j \leq N$, appear in the above inequality. We are also assuming that $r_{N+1} = 0$. Let us define the following function restricted to the set S such that $0 < r_N \leq \dots \leq r_{j_1} \leq r_1 = 1$:

$$\begin{aligned} \Psi(1, r_{j_1}, \dots, r_N) &= 1 + \log\left(\frac{1}{r_{j_1}}\right) + \sum_{j=j_1}^N \left(\prod_{k=j_1}^j r_k^{-\alpha_k} \right) \frac{r_j^{A_j-1} - r_{j+1}^{A_j-1}}{A_j - 1} \\ &\quad - \frac{p'}{p!} \int_0^1 \log\left(\frac{1}{x}\right) \prod_{j=j_1}^N \log^{\alpha_j}\left(\frac{1}{r_j x}\right) dx. \end{aligned}$$

Our goal is to prove that $\Psi(1, r_{j_1}, \dots, r_N) \leq 0$ on S . We proceed in the same recursive way as before, proving that Ψ is an increasing function on S , first in the variable r_N and hence reducing the number of variables, since the role of r_N is assumed by r_{N-1} . The process continues until we are restricted to the study of the function defined on just one variable $0 < r_{j_1} \leq 1$:

$$\begin{aligned} \Psi(1, r_{j_1}, \dots, r_{j_1}) &= 1 + \log\left(\frac{1}{r_{j_1}}\right) + \frac{1}{p-1} \\ &\quad - \frac{p'}{p!} \int_0^1 \log\left(\frac{1}{x}\right) \prod_{j=j_1}^N \log^{\alpha_j}\left(\frac{1}{r_{j_1} x}\right) dx \\ &= p' + \log\left(\frac{1}{r_{j_1}}\right) - \frac{p'}{p!} \int_0^1 \log\left(\frac{1}{x}\right) \log^{p-1}\left(\frac{1}{r_{j_1} x}\right) dx. \end{aligned}$$

Its derivative is

$$\begin{aligned} \Psi'(1, r_{j_1}, \dots, r_{j_1}) &= -\frac{1}{r_{j_1}} + \frac{1}{(p-1)! r_{j_1}} \int_0^1 \log\left(\frac{1}{x}\right) \log^{p-2}\left(\frac{1}{r_{j_1} x}\right) dx \\ &\geq -\frac{1}{r_{j_1}} + \frac{1}{(p-1)! r_{j_1}} \int_0^1 \log^{p-1}\left(\frac{1}{x}\right) dx = 0. \end{aligned}$$

Since $\Psi(1, \dots, 1) = 0$, we conclude that the maximum in the set S of the function Ψ is attained at point $(1, \dots, 1)$, and the sharp estimate (19) is obtained. \square

In Section 5 we will make some comments about the optimality of the constant D , on Theorem 3.3, and the remaining cases regarding C and F (see Table 1).

4. ENDPOINT ESTIMATES

We start the study of the endpoint estimates for the case $p = \infty$.

Proposition 4.1. *Let $f \in L^\infty(\mathbb{R}^+)$, then following sharp estimates hold:*

(i) *If f is a decreasing function,*

$$\|Sf - f\|_\infty \leq \|f\|_\infty.$$

Thus, $A = 1$.

(ii) *If f is a positive function,*

$$\|Sf - f\|_\infty \leq \|f\|_\infty.$$

Thus, $B = 1$.

(iii) *If f is a general function,*

$$(21) \quad \|Sf - f\|_\infty \leq 2\|f\|_\infty.$$

Thus, $C = 2$ (see [2, equation (5.5)]).

Proof. Let us first prove part (i). Since $|Sf(x) - f(x)| = Sf(x) - f(x) \leq Sf(x)$, then

$$\|Sf - f\|_\infty \leq \|Sf\|_\infty \leq \|f\|_\infty.$$

In order to see that this estimate is sharp, we just consider the decreasing function $f(x) = \chi_{(0,1)}(x)$, for which

$$Sf(x) - f(x) = \frac{1}{x}\chi_{(1,\infty)}(x).$$

Thus,

$$\|Sf - f\|_\infty = \|f\|_\infty = 1.$$

To prove part (ii), we observe that for every positive function f ,

$$|Sf(x) - f(x)| \leq \max\{Sf(x), f(x)\}.$$

Then

$$\|Sf - f\|_\infty \leq \max\{\|Sf\|_\infty, \|f\|_\infty\} \leq \|f\|_\infty.$$

Also the sharpness in this last estimate follows by considering the same characteristic function as in part (i).

Finally, for the proof of part (iii), we observe that trivially,

$$\|Sf - f\|_\infty \leq \|Sf\|_\infty + \|f\|_\infty \leq 2\|f\|_\infty.$$

The fact that the constant 2 is sharp in the inequality above follows by considering $f(x) = \chi_{(0,1)}(x) - \chi_{(1,2)}(x)$, for which $Sf(x) - f(x) = \frac{2}{x}\chi_{(1,2)}(x)$ and hence

$$\|Sf - f\|_\infty = 2 = 2\|f\|_\infty.$$

□

We now deal with the remaining constants relating S and S^* , in the case $p = \infty$. Observe that since, for any decreasing function f , $\|S^*f\|_\infty = \infty$, then $D = 0$ and we just need to consider the cases of positive and general functions, constants E and F , respectively.

Proposition 4.2. *Let $f \in L^\infty(\mathbb{R}^+)$, then following sharp estimates hold:*

(i) *If f is a positive function,*

$$\|Sf\|_\infty \leq \|S^*f\|_\infty.$$

Thus, $E = 1$.

(ii) *If f is a general function,*

$$\|Sf\|_\infty \leq 2\|S^*f\|_\infty.$$

Thus, $F = 2$.

Proof. Part (i) easily follows since, for a fixed $t > 0$,

$$\|S^*f\|_\infty = \int_0^\infty f(s) \frac{ds}{s} \geq \int_0^t f(s) \frac{ds}{s} \geq Sf(t).$$

The optimality of the inequality is due to the fact that, for $f_r(x) = \chi_{(r,r+1)}(x)$,

$$\|Sf_r\|_\infty = \frac{1}{r+1} \quad \text{and} \quad \|S^*f_r\|_\infty = \log\left(\frac{r+1}{r}\right).$$

Thus

$$\lim_{r \rightarrow \infty} \frac{\|Sf_r\|_\infty}{\|S^*f_r\|_\infty} = 1.$$

Part (ii) is a consequence of Proposition 4.1 (iii) and Lemma 1.2. The sharpness in the inequality follows by considering, for $\varepsilon > 0$,

$$f_\varepsilon(x) = \frac{x}{\varepsilon} \left(\chi_{(1-\varepsilon, 1+\varepsilon)}(x) - \chi_{(2-\varepsilon, 2)}(x) \right).$$

Then

$$S^*f_\varepsilon(x) = \chi_{(0, 1-\varepsilon)}(x) + \frac{1-x}{\varepsilon} \chi_{(1-\varepsilon, 1+\varepsilon)}(x) - \chi_{(1+\varepsilon, 2-\varepsilon)}(x) + \frac{x-2}{\varepsilon} \chi_{(2-\varepsilon, 2)}(x).$$

Thus, for every $\varepsilon > 0$, $\|S^*f_\varepsilon\|_\infty = 1$, and for $1 + \varepsilon < x < 2 - \varepsilon$, we have that $Sf_\varepsilon(x) = 2/x$. Hence,

$$\|Sf_\varepsilon\|_\infty \geq \frac{2}{1+\varepsilon}$$

and

$$2 = \lim_{\varepsilon \rightarrow 0} \frac{2}{1+\varepsilon} \leq \liminf_{\varepsilon \rightarrow 0^+} \frac{\|Sf_\varepsilon\|_\infty}{\|S^*f_\varepsilon\|_\infty}.$$

□

Continuing with the study of the endpoint cases, we now address what happens for $p = 1$. We start with the action of $S - \text{Id}$ on $L^1(\mathbb{R}^+)$ to obtain weak-type estimates. The sharp bound $1/\log 2$, for any $f \in L^1(\mathbb{R}^+)$, was obtained in [2] (see also [3] for further extensions to weak-type L^p estimates, $1 \leq p \leq 2$). In part (iii) below we will give a simpler argument to prove optimality, which will be useful in the proof of Proposition 4.4 (iii).

Proposition 4.3. *Let $f \in L^1(\mathbb{R}^+)$, then following sharp estimates hold:*

(i) *If f is a decreasing function,*

$$\|Sf - f\|_{1,\infty} \leq \|f\|_1.$$

Thus, $A = 1$.

(ii) If f is a positive function,

$$\|Sf - f\|_{1,\infty} \leq \|f\|_1.$$

Thus, $B = 1$.

(iii) If f is a general function,

$$\|Sf - f\|_{1,\infty} \leq \frac{1}{\log 2} \|f\|_1.$$

Thus, $C = 2$ (see [2, Theorem 3.3]).

Proof. To prove parts (i) and (ii), let us consider $f \geq 0$ and $t > 0$, and define $\alpha_t = \sup\{x > 0 : Sf(x) \geq t\}$. Thus $Sf(x) < t$, if $x > \alpha_t$. Observe that, since $f \in L^1(\mathbb{R}^+)$, $\lim_{x \rightarrow \infty} Sf(x) = 0$, and $\alpha_t < \infty$. Also,

$$|Sf(x) - f(x)| \leq \max\{f(x), Sf(x)\}.$$

Thus,

$$\begin{aligned} |\{x > 0 : |Sf(x) - f(x)| > t\}| &= |\{x \leq \alpha_t : |Sf(x) - f(x)| > t\}| \\ &\quad + |\{x > \alpha_t : |Sf(x) - f(x)| > t\}| \\ &\leq \alpha_t + |\{x > \alpha_t : f(x) > t\}| \\ &\leq \frac{1}{t} \int_0^{\alpha_t} f(x) dx + \frac{1}{t} \int_{\alpha_t}^{\infty} f(x) dx = \frac{1}{t} \|f\|_1. \end{aligned}$$

Therefore, for every $f \geq 0$

$$(22) \quad \|Sf - f\|_{1,\infty} \leq \|f\|_1.$$

Since for the decreasing function $g(x) = \chi_{(0,1)}(x)$, $\|g\|_1 = 1$, $Sg(x) - g(x) = \frac{1}{x}\chi_{(1,\infty)}(x)$, we get

$$\|Sg - g\|_{1,\infty} = \sup_{0 < \lambda \leq 1} \lambda \frac{1 - \lambda}{\lambda} = 1.$$

This proves that (22) is optimal, for both the cone of decreasing functions and the cone of positive functions.

Part (iii) is a consequence of [2, Theorem 3.3]. The optimality of the constant can be obtained by considering, for any $k \in \mathbb{N}$, the family of functions defined as

$$(23) \quad g_k(x) = k\chi_{(0,1/k)}(x) - \log x \chi_{(1,2)}(x).$$

Moreover, for this function $\|g_k\|_1 = 2 \log 2$, and

$$Sg_k(x) - g_k(x) = \frac{1}{x}\chi_{(1/k,1)}(x) + \chi_{(1,2)}(x) + \frac{2 - \log 4}{x}\chi_{(2,\infty)}(x).$$

Hence,

$$|\{x > 0 : |Sg_k(x) - g_k(x)| \geq 1\}| = 2 - \frac{1}{k}.$$

We obtain as a consequence,

$$(24) \quad \sup_f \frac{\|Sf - f\|_{1,\infty}}{\|f\|_1} \geq \lim_{k \rightarrow \infty} \frac{\|Sg_k - g_k\|_{1,\infty}}{\|g_k\|_1} \geq \lim_{k \rightarrow \infty} \frac{2 - 1/k}{2 \log 2} = \frac{1}{\log 2},$$

and, therefore, the constant $1/\log 2$ is sharp. \square

Proposition 4.4. *Let $f \in L^1(\mathbb{R}^+)$, then following sharp estimates hold:*

(i) *If f is a decreasing function,*

$$\|Sf\|_{1,\infty} \leq \|S^*f\|_1.$$

Thus, $D = 1$.

(ii) *If f is a positive function,*

$$\|Sf\|_{1,\infty} \leq \|S^*f\|_1.$$

Thus, $E = 1$.

(iii) *If f is a general function,*

$$\|Sf\|_{1,\infty} \leq \frac{1}{\log 2} \|S^*f\|_1.$$

Thus, $F = 1/\log 2$.

Proof. First of all we observe that, if $f \geq 0$, Fubini's theorem implies

$$\|S^*f\|_1 = \|f\|_1.$$

Hence, sharp weak-type estimates of parts (i) and (ii) follow from the corresponding well-known weak-type optimal estimate for the Hardy operator,

$$\|Sf\|_{1,\infty} \leq \|S^*f\|_1 = \|f\|_1.$$

As before, part (iii) is a consequence of Proposition 4.3 (iii) and Lemma 1.2. For the optimality, if g_k is as in (23), given $j, k \in \mathbb{N}$, we choose $g_{j,k}$ such that $g_{j,k} \rightarrow g_k$ in $L^1(\mathbb{R}^+)$ and there exists $f_{j,k}$ satisfying $S^*(f_{j,k}) = g_{j,k}$ (observe that, necessarily, $g_{j,k}$ is a continuous function). For example, it suffices to consider

$$f_{j,k}(x) = jkx \chi_{(1/k, 1/k+1/j)}(x) + \chi_{(1,2)}(x) - jx \log 2 \chi_{(2,2+1/j)}(x).$$

Thus, $Sf_{j,k} = Sg_{j,k} - g_{j,k}$, $S^*f_{j,k} = g_{j,k}$ and using (24)

$$F = \sup_f \frac{\|Sf\|_{1,\infty}}{\|S^*f\|_1} \geq \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \frac{\|Sg_{j,k} - g_{j,k}\|_{1,\infty}}{\|g_{j,k}\|_1} = \lim_{k \rightarrow \infty} \frac{\|Sg_k - g_k\|_{1,\infty}}{\|g_k\|_1} \geq \frac{1}{\log 2}.$$

□

5. FURTHER COMMENTS

In previous sections we have obtained all sharp constants of Table 1, A, \dots, F , for the range $1 < p \leq 2$ and the endpoints $p = 1, \infty$. For the remaining case $2 < p < \infty$, we do not know the values of A, B , and E . Let us review what can we say for the other cases:

Remark 5.1. Theorem 3.3 shows that for $p \geq 2$, an integer exponent, and f a decreasing function in $L^p(\mathbb{R}^+)$, the following inequality is sharp

$$(25) \quad \|Sf\|_p \leq \left(\frac{p'}{\Gamma(p+1)} \right)^{1/p} \|S^*f\|_p;$$

i.e., $D = \left(\frac{p'}{\Gamma(p+1)} \right)^{1/p}$, if $p = 2, 3, 4, \dots$. Now, checking the inequality (25) with $f(x) = \chi_{(0,1)}(x)$, we also get that $D \geq \left(\frac{p'}{\Gamma(p+1)} \right)^{1/p}$, for all values of $p \geq 2$. Hence, it is natural to conjecture that the estimate (25) holds and is sharp, for any $p \geq 2$.

Up to now, the best we can say for a general $p \geq 2$ is the following: recall that if D is the sharp constant in the inequality,

$$\|Sf\|_p \leq D\|S^*f\|_p,$$

valid for any decreasing f , then from the estimate (9) and the previous observation, we have

$$\left(\frac{p'}{\Gamma(p+1)}\right)^{1/p} \leq D \leq \frac{1}{(p-1)^{1/p}}.$$

In order to give a better bound, we observe that for every decreasing function f , $r > 1$, and $x > 0$

$$\int_x^\infty \frac{f(t)}{t} dx \geq f(rx) \log r.$$

Thus,

$$\|S^*f\|_p^p \geq \log^p r \int_0^\infty f^p(rx) dx = \frac{\log^p r}{r} \|f\|_p^p.$$

Optimizing this last inequality, we observe that since the maximum of $g(r) = \log^p r / r$ is attained at $r = e^p$, the following inequality holds

$$\|S^*f\|_p \geq \frac{p}{e} \|f\|_p.$$

From here, using (2), it follows that for every decreasing function,

$$\frac{\|Sf\|_p}{\|S^*f\|_p} \leq \frac{p' \|f\|_p}{\frac{p}{e} \|f\|_p} = \frac{e}{p-1}.$$

Therefore,

$$(26) \quad \left(\frac{p'}{\Gamma(p+1)}\right)^{1/p} \leq D \leq \min\left(\frac{e}{p-1}, \frac{1}{(p-1)^{1/p}}\right).$$

Finally, observe that

$$\lim_{p \rightarrow 2^+} (p-1) \left(\frac{p'}{\Gamma(p+1)}\right)^{1/p} = \lim_{p \rightarrow 2^+} (p-1) \min\left(\frac{e}{p-1}, \frac{1}{(p-1)^{1/p}}\right) = 1$$

and

$$\lim_{p \rightarrow \infty} (p-1) \left(\frac{p'}{\Gamma(p+1)}\right)^{1/p} = \lim_{p \rightarrow \infty} (p-1) \min\left(\frac{e}{p-1}, \frac{1}{(p-1)^{1/p}}\right) = e.$$

Hence, the inequalities in (26) are asymptotically optimal, for $p \geq 2$.

Remark 5.2. For general functions on the range $2 < p < \infty$ (i.e., the constants C and F), we do not really have any conjecture about the possible values (see also [2]). However, using Lemma 1.2 we can prove that $F \leq C$.

Regarding C , if we interpolate the isometric property with $p = 2$ and (21) for $p = \infty$, and evaluating at $f(x) = \chi_{(0,1)}(x) - \chi_{(1,2)}(x)$, we obtain that

$$2^{1-2/p} \geq C \geq \left(\frac{2^{p-1} - 1}{p-1}\right)^{1/p} > 1 = B, \quad p > 2.$$

Finally, observe that

$$\lim_{p \rightarrow 2^+} 2^{1-2/p} = \lim_{p \rightarrow 2^+} \left(\frac{2^{p-1} - 1}{p-1}\right)^{1/p} = 1$$

and

$$\lim_{p \rightarrow \infty} 2^{1-2/p} = \lim_{p \rightarrow \infty} \left(\frac{2^{p-1} - 1}{p - 1} \right)^{1/p} = 2.$$

Using these estimates and as we did in the proof of Proposition 4.4 (iii), we can also obtain a similar result for F :

$$2^{1-2/p} \geq C \geq F \geq \left(\frac{2^{p-1} - 1}{p - 1} \right)^{1/p} > 1 > \frac{1}{(p - 1)^{1/p}} = E, \quad p > 2.$$

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