$L \log \log L$ VERSIONS OF STEIN'S AND ZYGMUND'S THEOREMS FOR THE HARDY SPACE $H^{\log}(\mathbb{R}^d)$

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Abstract. We obtain versions of some classical results of Zygmund and Stein for functions belonging to the Hardy space $H^{\log}(\mathbb{R}^d)$ introduced by Bonami, Grellier, and Ky. We present further applications in the context of more general Orlicz spaces.

1. Introduction

The Hardy-Littlewood maximal function is a fundamental object in harmonic analysis, defined for a locally integrable function $f\colon \mathbb{R}^d \to \mathbb{C}$ by setting

$$M(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy \quad x \in \mathbb{R}^d,$$

where $B(x,r)$ denotes the open ball in $\mathbb{R}^d$ centered at x with radius $r > 0$ and $|A|$ denotes the Lebesgue measure of $A \subseteq \mathbb{R}^d$. It is a basic fact that the mapping $f \mapsto M(f)$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$. The maximal operator is also bounded from $L^1(\mathbb{R}^d)$ to weak-L^1, but does not map $L^1(\mathbb{R}^d)$ to itself (see, for instance, [10] for an in-depth discussion).

However, $M(f)$ is locally integrable provided f is compactly supported and satisfies the $L \log L$ condition

$$\int_{\mathbb{R}^d} |f(x)| \log^+ |f(x)| dx < \infty,$$

where, as usual, $\log^+ |x| = \max\{\log |x|, 0\}$. In a 1969 paper, E.M. Stein [8] proved that this $L \log L$ condition is both sufficient and necessary for integrability of the Hardy-Littlewood maximal function, in the following sense: if f is supported in some finite ball $B = B(r)$ of radius $0 < r < \infty$, then

$$\int_B M(f)dx < \infty \quad \text{if, and only if,} \quad \int_B |f(x)| \log^+ |f(x)| dx < \infty.$$

Another classical result that involves the space $L \log L$ is due to Zygmund, and asserts that the periodic Hilbert transform maps $L \log L(\mathbb{T})$ to $L^1(\mathbb{T})$; see e.g. Theorem 2.8 in Chapter VII of [12]. This implies that $L \log L(\mathbb{T})$ is contained in the real Hardy space $H^1(\mathbb{T})$ consisting of integrable functions on the torus whose Hilbert transforms are integrable. Moreover, as shown by Stein in [8], Zygmund's theorem has a partial converse, namely if $f \in H^1(\mathbb{T})$ and f is non-negative, then f necessarily belongs to $L \log L(\mathbb{T})$. Therefore, in view of the aforementioned results of Zygmund and Stein, the Hardy space $H^1(\mathbb{T})$ is, in terms of magnitude, associated with the Orlicz space $L \log L(\mathbb{T})$.

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In this note, we obtain versions of these results for the function space $H^{\log}(\mathbb{R}^d)$ that was recently introduced by A. Bonami, S. Grellier, and L.D. Ky in [3]. To do this, we identify the correct analog of $L\log L$ in this context, which turns out to be $L\log\log L$: given a measurable subset B of $\mathbb{R}^d$, $L\log\log L(B)$ denotes the class of all locally integrable functions f with $\operatorname{supp}(f) \subseteq B$ and

$$\int_B |f(x)| \log^+ \log^+ |f(x)| dx < \infty.$$

In order to formally state our results, we now give the definition of the space $H^{\log}(\mathbb{R}^d)$. Let $\Psi\colon \mathbb{R}^d \times [0,\infty) \to [0,\infty)$ denote the function given by

$$\Psi(x,t) := \frac{t}{\log(e+t) + \log(e+|x|)}, \quad (x,t) \in \mathbb{R}^d \times [0,\infty).$$

If B is a subset of $\mathbb{R}^d$, one defines $L_\Psi(B)$ to be the space of all locally integrable functions f on B satisfying

$$\int_B \Psi(x, |f(x)|)dx < \infty.$$

We shall also fix a non-negative function $\phi \in C^\infty(\mathbb{R}^d)$, which is supported in the unit ball of $\mathbb{R}^d$ and has $\int_{\mathbb{R}^d} \phi(y)dy = 1$ and $\phi(x) = c_0$ for all $|x| \le 1/2$, where c_0 is a constant. Given an $\epsilon > 0$, we use the standard notation $\phi_\epsilon(x) := \epsilon^{-d}\phi(\epsilon^{-1}x)$, $x \in \mathbb{R}^d$.

Definition ($H^{\log}$, see [3, 11])**.** *If ϕ is as above, consider the maximal function*

$$M_\phi(f)(x) := \sup_{\epsilon>0} |(f * \phi_\epsilon)(x)|, \quad x \in \mathbb{R}^d.$$

The Hardy space $H^{\log}(\mathbb{R}^d)$ is defined to be the space of all locally integrable functions f on $\mathbb{R}^d$ such that $M_\phi(f) \in L_\Psi(\mathbb{R}^d)$, that is, $M_\phi(f)$ satisfies

$$\int_{\mathbb{R}^d} \Psi(x, M_\phi(f)(x))dx < \infty.$$

The motivation for defining the space $H^{\log}$ comes from the study of products of functions in the real Hardy space $H^1(\mathbb{R}^d)$ and functions in $\mathrm{BMO}(\mathbb{R}^d)$, the class of functions of bounded mean oscillation. Following earlier work by Bonami, T. Iwaniec, P. Jones, and M. Zinsmeister in [2], it was shown by Bonami, Grellier, and Ky [3] that the product fg, in the sense of distributions, of a function $f \in H^1(\mathbb{R}^d)$ and a function $g \in \mathrm{BMO}(\mathbb{R}^d)$ can be represented as a sum of a continuous bilinear mapping into $L^1(\mathbb{R}^d)$ and a continuous bilinear operator into $H^{\log}(\mathbb{R}^d)$.

Here is our version of Stein's lemma for L_Ψ.

Theorem 1. *Let f be a measurable function supported in a closed ball $B \subsetneq \mathbb{R}^d$. Then $M(f) \in L_\Psi(B)$ if, and only if, $f \in L\log\log L(B)$.*

Our proof in fact leads to a more general version of Theorem 1. We discuss this, and give a proof of Theorem 1 in Section 2.

Next is the analog of Zygmund's result for $H^{\log}(\mathbb{R}^d)$.

Theorem 2. *Let B denote the closed unit ball in $\mathbb{R}^d$. If $f \in L\log\log L(B)$ and $\int_B f(y)dy = 0$, then $f \in H^{\log}(\mathbb{R}^d)$.*

We remark that the mean-zero condition in the hypothesis is in fact necessary in order to place a compactly supported function in $H^{\log}$. The proof of Theorem 2 is presented in Section 3.

In Section 4, we discuss further extensions to the periodic setting.

2. Proof of the Stein-type Theorem for L_Ψ and further extensions

We begin with an elementary observation that will be implicitly used several times in the sequel: if $\Phi : [0,\infty) \to [0,\infty)$ is an increasing function, then for every positive constant α_0 one has

$$\int_B \Phi(|g(x)|)dx \le \Phi(\alpha_0)|B| + \int_{\{|g|>\alpha_0\}} \Phi(|g(x)|)dx$$

for each measurable set B in $\mathbb{R}^d$ with finite measure.

We now turn to the proof of our first theorem.

Proof of Theorem 1. Assume first that $f \in L\log\log L(B)$. The main observation is that locally the space L_Ψ essentially coincides with the Orlicz space defined in terms of the function $\Psi_0(t) := t \cdot [\log(e+t)]^{-1}$, $t \ge 0$ and so, one can employ the arguments of Stein [8].

To be more precise, we note that for $x \in B$ one has

$$\log(e + M(f)(x)) \le \log\left((e+|x|)(e+M(f)(x))\right) \le c\log(e+M(f)(x)), \tag{2.1}$$

for a constant c that only depends on B. Next, an integration by parts yields

$$\int_e^y \frac{1}{\log\alpha}d\alpha = \frac{y}{\log y} - e + \int_e^y \frac{1}{\log^2\alpha}d\alpha,$$

so that

$$\frac{y}{\log y} \le e + \int_e^y \frac{1}{\log\alpha}d\alpha, \quad \text{for} \quad y > e.$$

Together, these two observations imply that

$$\begin{aligned}\int_B \Psi(x, M(f)(x))dx &\lesssim_B 1 + \int_{B\cap\{M(f)>e\}} \left(\int_e^{M(f)(x)} \frac{1}{\log\alpha}d\alpha\right)dx \\ &= 1 + \int_e^\infty \frac{1}{\log\alpha}\cdot |\{x\in B : M(f)(x) > \alpha\}|d\alpha.\end{aligned}$$

To estimate the last integral, note that there exists an absolute constant $C_d > 0$ such that

$$|\{x \in \mathbb{R}^d : M(f)(x) > \alpha\}| \le \frac{C_d}{\alpha}\int_{\{|f|>\alpha/2\}} |f(x)|dx \tag{2.2}$$

for all $\alpha > 0$; see e.g. [8, (5)] or Section 5.2 (a) in Chapter I in [9]. We thus deduce from (2.2) that

$$\begin{aligned}\int_B \Psi(x, M(f)(x))dx &\lesssim_B 1 + \int_B |f(x)|\cdot\left(\int_e^{2|f(x)|} \frac{1}{\alpha\log\alpha}d\alpha\right)dx \\ &\lesssim 1 + \int_B |f(x)|\log^+\log^+|f(x)|dx,\end{aligned}$$

which implies that $M(f) \in L_\Psi(B)$.

To prove the reverse implication, assume that for some f supported in B with $f \in L^1(B)$ we have $M(f) \in L_\Psi(B)$. Our task is to show that $f \in L\log\log L(B)$. In order to accomplish this, we shall make use of the fact that there exists a $\rho > 2$, depending only on $\|f\|_{L^1(B)}$ and B, such that we also have $M(f) \in L_\Psi(\rho B)$ and moreover, for every $\alpha \ge e^e$,

$$|\{x \in \rho B : M(f)(x) > c_1\cdot\alpha\}| \ge \frac{c_2}{\alpha}\int_{B\cap\{|f|>\alpha\}} |f(x)|dx, \tag{2.3}$$

where c_1, c_2 are positive constants that can be taken to be independent of f and α. Indeed, arguing as in the proof of [8, Lemma 1], note that for every $r > 2$ one has

$$M(f)(x) \lesssim \frac{1}{(r-1)^d|B|} \cdot \|f\|_{L^1(B)} \quad \text{for all } x \in \mathbb{R}^d \setminus rB. \tag{2.4}$$

Hence, if we choose $\rho > 2$ to be large enough, then $M(f)(x) < e^e \leq \alpha$ for all $x \in \mathbb{R}^d \setminus \rho B$ and so, (2.3) follows from [8, Inequality (6)].

Furthermore, one can check that $M(f) \in L_\Psi(\rho B)$. Indeed, if we write $B = B(x_0, r_0)$ then, as in [8], it follows from the definition of M and the fact that $\operatorname{supp}(f) \subseteq B$ that there exists a constant $c_0 > 0$, depending only on the dimension, such that for every $x \in 2B \setminus B$ one has

$$M(f)(x) \leq c_0 \cdot M(f)\left(x_0 + r_0^2 \cdot \frac{x - x_0}{|x - x_0|^2}\right) \tag{2.5}$$

and so, $M(f) \in L_\Psi(2B)$. To show that (2.5) implies that $M(f) \in L_\Psi(B)$, observe first that the function $\Psi_0(s) = s/\log(e+s)$ is increasing on $[0, +\infty)$, and for all $t \geq 1$ and all $s > 0$,

$$1 \geq \frac{\log(e+s)}{\log(e+ts)} = \frac{\log(e+s)}{\log(e/t+s) + \log t} \geq \frac{\log(e+s)}{\log(e+s) + \log t} \geq \frac{1}{1+\log t},$$

so Ψ_0 satisfies

$$t(1+\log t)^{-1}\Psi_0(s) \leq \Psi_0(st) \leq t\Psi_0(s), \tag{2.6}$$

which implies that for all $c > 0$ and all $s > 0$

$$\Psi_0(cs) \sim_c \Psi_0(s).$$

Observe that a change to polar coordinates, followed by another a change of variables and elementary estimates yield

$$\begin{aligned}
\int_{2B \setminus B} \Psi_0(Mf(x))dx &\lesssim \int_{r_0}^{2r_0} s^{d-1} \int_{S^{d-1}} \Psi_0(Mf(x_0 + r_0^2\theta/s))d\sigma(\theta)ds \\
&\sim r_0^{-1} \int_{\frac{1}{2}}^{1} t^{-1-d} \int_{S^{d-1}} \Psi_0(Mf(x_0 + r_0 t\theta))d\sigma(\theta)dt \\
&\sim \int_{\frac{1}{2}}^{1} t^{d-1} \int_{S^{d-1}} \Psi_0(Mf(x_0 + r_0 t\theta))d\sigma(\theta)dt \\
&\lesssim \int_B \Psi(x, Mf(x))dx.
\end{aligned}$$

Moreover, we deduce from (2.4) that $M(f)$ belongs to $L_\Psi(\rho B \setminus 2B)$ and it thus follows that $M(f) \in L_\Psi(\rho B)$, as desired.

Next, note that by the same reasoning as in the proof of sufficiency and by Fubini's theorem,

$$\begin{aligned}
\int_{\rho B} \Psi(x, M(f)(x))dx &\gtrsim \int_{\rho B \cap \{M(f) > \max\{e^e, |x_0| + r_0\}\}} \frac{M(f)(x)}{\log(M(f)(x))} dx \\
&\gtrsim \int_{\rho B \cap \{M(f) > \max\{e^e, |x_0| + r_0\}\}} \left(\int_{e^e}^{M(f)(x)} \frac{1}{\log \alpha} d\alpha\right) dx \\
&\gtrsim \int_{\max\{e^e, |x_0| + r_0\}}^{\infty} \frac{1}{\log \alpha} \cdot |\{x \in \rho B : M(f)(x) > c_2 \cdot \alpha\}| d\alpha.
\end{aligned}$$

By using (2.3), we now get

$$\infty > \int_{\rho B} \Psi(x, M(f)(x))dx \gtrsim \int_B |f(x)| \cdot \left(\int_{\max\{e^e, |x_0|+r_0\}}^{|f(x)|} \frac{1}{\alpha \log \alpha} d\alpha \right) dx$$
$$\gtrsim 1 + \int_B |f(x)| \log^+ \log^+ |f(x)| dx$$

and this completes the proof of Theorem 1. □

Remark 3. *Let B_0 denote the closed unit ball in $\mathbb{R}^d$. Given a small $\delta \in (0, e^{-e})$, if, as on pp. 58–59 in [5], one considers $f := \delta^{-d}\chi_{\{|x|<\delta\}}$ then $M(f)(x) \sim |x|^{-d}$ for all $|x| > 2\delta$ and so,*

$$\int_{B_0} |f(x)| \log^+ \log^+ |f(x)| dx \sim \log(\log(\delta^{-1})) \sim \int_{B_0} \Psi(x, M(f)(x))dx. \tag{2.7}$$

This shows that given $L_\Psi(B_0)$, the space $L \log \log L(B_0)$ in the statement of Theorem 1 is best possible in general, in terms of size.

Indeed, the left-hand side of (2.7) follows by direct calculation. On the other hand, using (2.1), (2.6), a change to polar coordinates, and further change of variables yield

$$\int_{B_0} \Psi(x, M(f)(x))dx \sim 1 + \int_{2\delta}^1 \frac{1}{\log(e + s^{-d})} \frac{ds}{s}$$
$$\sim 1 + \int_1^{(2\delta)^{-1}} \frac{1}{\log(e + u^d)} \frac{du}{u} \sim 1 + \int_e^{(2\delta)^{-1}} \frac{1}{\log(u)} \frac{du}{u},$$

from where the right-hand side of (2.7) follows.

2.1. **Further generalizations.** Assume that $\Psi : \mathbb{R}^d \times [0, \infty)$ is a non-negative function satisfying the following properties:

(1) For every $x \in \mathbb{R}^d$ fixed, $\Psi(x, t) = \Psi_x(t)$ is *Orlicz* in $t \in [0, \infty)$, namely $\Psi_x(0) = 0$, Ψ_x is increasing and convex on $[0, \infty)$ with $\Psi_x(t) > 0$ for all $t > 0$ and $\Psi_x(t) \to \infty$ as $t \to \infty$.
   Moreover, assume that there exists an absolute constant $C_0 > 0$ such that $\Psi_x(2t) \leq C_0 \Psi_x(t)$ for all $x \in \mathbb{R}^d$ and every $t \in [0, \infty)$.
(2) If K is a compact set in $\mathbb{R}^d$, then there exist $x_1, x_2 \in K$ and a constant $C_K > 0$ such that
$$C_K^{-1} < \Psi(x_1, t) \leq \Psi(x, t) \leq \Psi(x_2, t) < C_K$$
   for every $x \in K$ and for all $t > 0$.
(3) If we write $\Psi(x, t) = \Psi_x(t) = \int_0^t \psi_x(s)ds$, then for every α_0, β_0 with $0 < \alpha_0 < \beta_0$ one has
$$\int_{\alpha_0}^{\beta_0} \frac{\psi_x(s)}{s} ds < \infty$$
   for every $x \in \mathbb{R}^d$.

By carefully examining the proof of Theorem 1, one obtains the following result.

Theorem 4. *Let $\Psi(x, t) = \int_0^t \psi_x(s)ds$, $(x, t) \in \mathbb{R}^d \times [0, \infty)$, be as above.*

Fix a closed ball B with $B \subsetneq \mathbb{R}^d$ and let f be such that supp$(f) \subseteq B$. *Then, $M(f) \in L_\Psi(B)$ if, and only if,*

$$\int_{\{|f|>\alpha_0\}} |f(x)| \cdot \left(\int_{\alpha_0}^{|f(x)|} \frac{\psi_x(s)}{s} ds \right) dx < \infty$$

for every $\alpha_0 > 0$.

Theorem 4 applies to certain Orlicz spaces considered in connection with convergence of Fourier series, see e.g. [1, 7], and the recent paper by V. Lie [6]; we give some sample applications in Subsection 4.1.

3. Proof of the Zygmund-type Theorem for $H^{\log}(\mathbb{R}^d)$

We begin with the following elementary lemmas.

Lemma 5. *Consider the function* $g : [0,\infty)^2 \to [0,\infty)$ *given by*

$$g(s,t) := \frac{1}{\log(e+t)+\log(e+s)}, \quad (s,t) \in [0,\infty)^2.$$

Then one has

$$\Psi(x,t) \leq \int_0^t g(|x|,\tau)d\tau \leq 2\Psi(x,t)$$

for all $(x,t) \in \mathbb{R}^d \times [0,\infty)$.

Proof. The function $t \mapsto g(s,t) = \frac{1}{\ln((e+t)(e+s))}$ is decreasing, so clearly

$$\int_0^t g(|x|,s)ds \geq tg(|x|,t) = \Psi(x,t).$$

We now address the upper bound. A calculation yields that

$$\partial_t(t^\epsilon g(|x|,t)) = \frac{t^\epsilon}{\ln(e+t)+\ln(1+|x|)}\left(\frac{\epsilon}{t} - \frac{1}{(e+t)(\ln(e+t)+\ln(e+|x|)}\right),$$

and we observe that the term within the parenthesis is positive if, and only if,

$$\frac{\epsilon}{t} - \frac{1}{(e+t)(\ln(e+t)+\ln(e+|x|)} > 0,$$

which for $\epsilon = \frac{1}{2}$ is equivalent to the inequality

$$(e+t)(\ln(e+t)+\ln(e+|x|)) > 2t.$$

But clearly

$$(e+t)(\ln(e+t)+\ln(e+|x|)) \geq 2(e+t) > 2t.$$

Thus $s \mapsto s^\epsilon g(|x|,s)$ is increasing for $\epsilon = 1/2$, which implies that

$$\int_0^t g(|x|,s)ds = \int_0^t s^{-\epsilon}s^\epsilon g(|x|,s)ds \leq \frac{1}{1-\epsilon}\Psi(x,t) = 2\Psi(x,t)$$

and this completes the proof of the lemma. □

Lemma 6. *Let* $x_0 \in \mathbb{R}^d$ *be fixed and define* $(\tau_{x_0}f)(x) := f(x+x_0)$.
Then $f \in H^{\log}(\mathbb{R}^d)$ *if, and only if,* $\tau_{x_0}f \in H^{\log}(\mathbb{R}^d)$.

Proof. Note that it suffices to prove that for any $x_0 \in \mathbb{R}^d$ and $f \in H^{\log}(\mathbb{R}^d)$ one also has that $\tau_{x_0}f \in H^{\log}(\mathbb{R}^d)$.

Towards this aim, fix an $x_0 \in \mathbb{R}^d$ and an $f \in H^{\log}(\mathbb{R}^d)$. Observe that, by using a change of variables and the translation invariance of M_ϕ, we may write

$$I := \int_{\mathbb{R}^d} \frac{M_\phi(\tau_{x_0}f)(x)}{\log(e+|x|)+\log(e+M_\phi(\tau_{x_0}f)(x))}dx$$

as

$$I = \int_{\mathbb{R}^d} \frac{M_\phi(f)(x)}{\log(e+|x-x_0|)+\log(e+M_\phi(f)(x))}dx.$$

To prove that $I < \infty$, we split

$$I = I_1 + I_2,$$

where

$$I_1 := \int_{|x|>4|x_0|} \frac{M_\phi(f)(x)}{\log(e+|x-x_0|)+\log(e+M_\phi(f)(x))}dx$$

and

$$I_2 := \int_{|x|\leq 4|x_0|} \frac{M_\phi(f)(x)}{\log(e+|x-x_0|)+\log(e+M_\phi(f)(x))}dx.$$

To show that $I_1 < \infty$, observe that for $|x| > 4|x_0|$ one has

$$\frac{4|x-x_0|}{5} < |x| < \frac{4|x-x_0|}{3}$$

and so,

$$\begin{aligned} I_1 &\lesssim \int_{|x|>4|x_0|} \frac{M_\phi(f)(x)}{\log(e+|x|)+\log(e+M_\phi(f)(x))}dx \\ &\leq \int_{\mathbb{R}^d} \frac{M_\phi(f)(x)}{\log(e+|x|)+\log(e+M_\phi(f)(x))}dx. \end{aligned}$$

Since $f \in H^{\log}(\mathbb{R}^d)$, the last integral is finite and we thus deduce that $I_1 < \infty$. Next, to show that $I_2 < \infty$, we have

$$\begin{aligned} I_2 &\leq \int_{|x|\leq 4|x_0|} \frac{M_\phi(f)(x)}{1+\log(e+M_\phi(f)(x))}dx \\ &\lesssim_{|x_0|} \int_{|x|\leq 4|x_0|} \frac{M_\phi(f)(x)}{\log(e+|x|)+\log(e+M_\phi(f)(x))}dx \\ &\leq \int_{\mathbb{R}^d} \frac{M_\phi(f)(x)}{\log(e+|x|)+\log(e+M_\phi(f)(x))}dx \end{aligned}$$

and so, $I_2 < \infty$, as $f \in H^{\log}(\mathbb{R}^d)$. Therefore, $I < \infty$ and it thus follows that $\tau_{x_0} f \in H^{\log}(\mathbb{R}^d)$. □

To obtain the desired variant of Zygmund's theorem, we shall use the fact that functions in $H^{\log}(\mathbb{R}^d)$ have mean zero.

Lemma 7. *If $f \in H^{\log}(\mathbb{R}^d)$ is compactly supported, then $\int_{\mathbb{R}^d} f(y)dy = 0$.*

Proof. Let f be a given function in $H^{\log}(\mathbb{R}^d)$ with compact support. In light of Lemma 6, we may assume, without loss of generality, that f is supported in a closed ball B_r centered at 0 with radius $r > 0$, i.e. $\text{supp}(f) \subseteq B_r := \{x \in \mathbb{R}^d : |x| \leq r\}$.

To prove the lemma, take an $x \in \mathbb{R}^d$ with $|x| > 2r$ and observe that, by the definition of ϕ_ϵ, we can take $\epsilon = 4|x|$ to get

$$|f * \phi_\epsilon(x)| = \frac{1}{\epsilon^d}\left|\int_{B_r} f(y)\phi\left(\frac{x-y}{\epsilon}\right)dy\right| \gtrsim \frac{1}{|x|^d}\cdot\left|\int_{B_r} f(y)dy\right|$$

as we then have $\phi(\epsilon^{-1}(x-y)) = c_0$ for $y \in B_r$. Therefore, for all $|x| > 2r$ and $\epsilon = 4|x|$, we have

$$M_\phi(f)(x) \gtrsim \frac{1}{|x|^d}\cdot\left|\int_{B_r} f(y)dy\right|$$

and so, we deduce from Lemma 5 that

$$\Psi(x, M_\phi(f)(x)) \gtrsim \frac{1}{|x|^d\log(e+|x|)}\cdot\left|\int_{B_r} f(y)dy\right|$$

for $|x|$ large enough.

Hence, if $\int f(y)dy \neq 0$, then the function $\Psi(x, M_\phi(f)(x))$ does not belong to $L^1(\mathbb{R}^d)$, which is a contradiction. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let B denote the unit closed ball in $\mathbb{R}^d$. Fix a function f with $\operatorname{supp}(f) \subseteq B$, $\int_B f(y)dy = 0$ and $f \in L \log\log L(B)$. First of all, observe that

$$M_\phi(f)(x) \lesssim M(f)(x) \quad \text{for all } x \in \mathbb{R}^d,$$

where $M(f)$ denotes the Hardy-Littlewood maximal function of f; see e.g. Theorem 2 on pp. 62–63 in [9]. We thus deduce from Lemma 5 that

$$\Psi(x, M_\phi(f)(x)) \lesssim \Psi(x, M(f)(x)) \quad \text{for all } x \in \mathbb{R}^d$$

and hence, by using Theorem 1, we obtain

$$\int_{2B} \Psi(x, M_\phi(f)(x))dx \lesssim 1 + \int_{2B} |f(x)| \log^+ \log^+ |f(x)|dx, \tag{3.1}$$

where $2B := \{x \in \mathbb{R}^d : |x| \leq 2\}$.

To estimate the integral of $\Psi(x, M_\phi(f)(x))$ for $x \in \mathbb{R}^d \setminus 2B$, we shall make use of the cancellation of f. To be more specific, observe that if $|x| > 2$ then for every $\epsilon < |x|/2$, one has that

$$f * \phi_\epsilon(x) = \frac{1}{\epsilon^d} \int_B f(y)\phi\Big(\frac{x-y}{\epsilon}\Big)dy = 0$$

since $|x-y|/\epsilon > 1$ whenever $y \in B$. Therefore, we may restrict ourselves to $\epsilon \geq |x|/2$ when $|x| > 2$. Hence, for $\epsilon \geq |x|/2$, by exploiting the cancellation of f and using a Lipschitz estimate on ϕ_ϵ, we obtain

$$\begin{aligned} |f * \phi_\epsilon(x)| &= \frac{1}{\epsilon^d}\left| \int_B f(y)\phi\Big(\frac{x-y}{\epsilon}\Big)dy \right| = \frac{1}{\epsilon^d}\left| \int_B f(y)\Big[\phi\Big(\frac{x-y}{\epsilon}\Big) - \phi\Big(\frac{x}{\epsilon}\Big)\Big]dy \right| \\ &\lesssim_\phi \frac{1}{\epsilon^{d+1}} \int_B |y \cdot f(y)|dy \lesssim \frac{1}{|x|^{d+1}}\left[1 + \int_B |f(y)| \log^+ \log^+ |f(y)|dy\right]. \end{aligned}$$

We thus deduce that, for every $x \in \mathbb{R}^d \setminus 2B$,

$$|M_\phi(f)(x)| \lesssim \frac{1}{|x|^{d+1}}\left[1 + \int_B |f(y)| \log^+ \log^+ |f(y)|dy\right]$$

and so,

$$\begin{aligned} &\int_{\mathbb{R}^d \setminus 2B} \Psi(x, M_\phi(x))dx \\ &\lesssim \left[1 + \int_B |f(y)| \log^+ \log^+ |f(y)|dy\right] \cdot \int_{\mathbb{R}^d \setminus 2B} \frac{1}{|x|^{d+1}\log(e + |x|)}dx \\ &\lesssim 1 + \int_B |f(y)| \log^+ \log^+ |f(y)|dy, \end{aligned}$$

as desired. Therefore, Theorem 2 is now established by using the last estimate combined with (3.1). □

3.1. **A partial converse.** As in the classical setting of the real Hardy space H^1, see [8], Theorem 2 has a partial converse. To be more precise, if $f \in H^{\log}(\mathbb{R}^d)$ and $f > 0$ on an open set U, then the function f belongs to $L \log\log L(K)$ for every compact set $K \subset U$.

Indeed, to see this, note that if f is as above then

$$M_\phi(f)(x) \gtrsim M(f \cdot \eta_K)(x) \quad \text{for all } x \in K,$$

where η_K is an appropriate Schwartz function with $\eta_K \sim 1$ on K; see e.g. Section 5.3 in Chapter III in [10]. Hence, by using Lemma 5 and Theorem 1, we get

$$\int_{\mathbb{R}^d} \Psi(x, M_\phi(f)(x))dx \geq \int_K \Psi(x, M_\phi(f)(x))dx \gtrsim \int_K \Psi(x, M(\eta_K \cdot f)(x))dx$$
$$\gtrsim 1 + \int_K |f(x)| \log^+ \log^+ |f(x)|dx.$$

4. Variants in the periodic setting

Following [2], define $H^{\log}(\mathbb{D})$ to be the space of all holomorphic functions F on the unit disk $\mathbb{D}$ of $\mathbb{C}$ such that

$$\sup_{0<r<1} \int_0^{2\pi} \frac{|F(re^{i\theta})|}{\log(e + |F(re^{i\theta})|)} d\theta < \infty.$$

For $0 < q \leq \infty$, let $H^q(\mathbb{D})$ denote the classical Hardy space on $\mathbb{D}$ consisting of analytic functions F having

$$\sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|^p d\theta < \infty;$$

see for instance [4]. Then

$$H^1(\mathbb{D}) \subsetneq H^{\log}(\mathbb{D}) \subsetneq H^p(\mathbb{D}) \quad \text{for all } 0 < p < 1.$$

Hence, if $F \in H^{\log}(\mathbb{D})$ then F has a non-tangential limit F^* at almost every point of $\mathbb{T} = \partial\mathbb{D}$, and this non-tangential limit lies in $L^p(\mathbb{T})$ for $0 < p < 1$. See [4, Theorem 2.2] for details. Moreover, by using [2, Proposition 8.2], one may identify $H^{\log}(\mathbb{D})$ with the space of all measurable functions f on the torus such that

$$\int_0^{2\pi} \Psi_0\Big(\sup_{0<r<1} |P_r * f(\theta)|\Big) d\theta < \infty,$$

where $\Psi_0(t) := t \cdot [\log(e + t)]^{-1}$ $(t \geq 0)$ and for $0 < r < 1$, $\theta \in [0, 2\pi)$,

$$P_r(\theta) := \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}$$

denotes the Poisson kernel in the unit disk.

There is a periodic version of Theorem 1, namely $M(f) \in L_{\Psi_0}(\mathbb{T})$ if, and only if, $f \in L \log\log L(\mathbb{T})$. Combining this with Lemma 5, one obtains the following result.

Proposition 8. *One has the inclusion*

$$L \log\log L(\mathbb{T}) \subseteq H^{\log}(\mathbb{T}).$$

Moreover, arguing as in the previous section and using the necessity in Theorem 1 as well as Proposition 8 and Lemma 5, one can show that if $f \in H^{\log}(\mathbb{T})$ and f is non-negative, then $f \in L \log\log L(\mathbb{T})$.

Proposition 9. *One has*

$$\{f \in L \log\log L(\mathbb{T}) : f \geq 0 \text{ a.e. on } \mathbb{T}\} = \{f \in H^{\log}(\mathbb{T}) : f \geq 0 \text{ a.e. on } \mathbb{T}\}.$$

Proof. Note that Proposition 8 implies that

$$\{f \in L \log\log L(\mathbb{T}) : f \geq 0 \text{ a.e. on } \mathbb{T}\} \subseteq \{f \in H^{\log}(\mathbb{T}) : f \geq 0 \text{ a.e. on } \mathbb{T}\}. \tag{4.1}$$

To prove the reverse inclusion, take a non-negative function $f \in H^{\log}(\mathbb{T})$ and notice that it follows from the work of Stein [8] that

$$|\{\theta \in \mathbb{T} : M(f)(\theta) > c_1\alpha\}| \geq \frac{c_2}{\alpha} \int_{\{|f|>\alpha\}} |f(\theta)| d\theta, \tag{4.2}$$

where $c_1, c_2 > 0$ are absolute constants. Hence, by arguing as in the proof of Theorem 1, it follows from (4.2) (noting that the periodic case is easier as one does not need to consider the contribution away from the support of f) that

$$\int_{\mathbb{T}} \Psi_0(M(f))(\theta)d\theta \gtrsim 1 + \int_{\mathbb{T}} |f(x)| \log^+ \log^+ |f(\theta)| d\theta. \tag{4.3}$$

Since $f \geq 0$ a.e. on $\mathbb{T}$, as in the Euclidean case, one has

$$\sup_{0<r<1} |P_r * f(\theta)| \gtrsim M(f)(\theta) \quad \text{for a.e. } \theta \in \mathbb{T}. \tag{4.4}$$

Hence, by using (4.3), (4.4) and Lemma 5, we deduce that $f \in L \log \log L(\mathbb{T})$ and so,

$$\{f \in H^{\log}(\mathbb{T}) : f \geq 0 \text{ a.e. on } \mathbb{T}\} \subseteq \{f \in L \log \log L(\mathbb{T}) : f \geq 0 \text{ a.e. on } \mathbb{T}\}. \tag{4.5}$$

The desired fact is a consequence of (4.1) and (4.5). □

4.1. **Some further applications.** We conclude with some applications of Theorem 4 in the periodic setting. The function

$$\Psi(x,t) = \Psi(t) = t \log^+ t \, \log^+ \log^+ t$$

appearing in [7] satisfies the hypotheses appearing in Theorem 4, and we now determine which space maps into L_Ψ via the maximal function. With the associated ψ defined as before, an integration by parts yields

$$\int \frac{\psi(s)}{s} ds = \frac{1}{2}(\log^+ s)^2 \log^+ \log^+ s + \log^+ s \log^+ \log^+ s - \frac{1}{4}(\log^+ s)^2.$$

This allows us to conclude that, for this choice of Ψ,

$$M(f) \in L_\Psi(\mathbb{T}) \quad \text{if, and only if,} \quad f \in L \log^2 L \log \log L(\mathbb{T}).$$

Turning to the space $L \log \log L \log \log \log \log L$ appearing in Lie's paper [6], we can check where the maximal operator maps this space. Performing the appropriate computations, we obtain that

$$\int_{\mathbb{T}} \frac{M(f)}{\log(M(f)+e)} \log^+ \log^+ \log^+ \log^+ M(f) dx < \infty$$

if, and only if,

$$f \in L \log \log L \log \log \log \log L(\mathbb{T}).$$

Roughly speaking, the contents of Theorem 4 and the computations presented above can be summarized as follows. Let Φ_0 be a given Orlicz function, namely $\Phi_0 : [0,\infty) \to [0,\infty)$ is an increasing, convex function with $\Phi_0(0) = 0$ and $\Phi_0(t) \to \infty$ as $t \to \infty$. Suppose that one can find non-negative, increasing functions M, S with

$$\Phi_0(t) = M(t) \cdot S(t) \quad (t > 0)$$

and such that, for $0 < \alpha < t$, one can easily compute

$$F_\alpha(t) := \int_\alpha^t \frac{M'(s)}{s} ds$$

in closed form and, moreover, that there exists an $\alpha_0 > 0$ with the property that for every $\alpha \geq \alpha_0$ one has

$$F_\alpha(t) \cdot S(t) \gtrsim \int_\alpha^t \Big(\frac{M(s)}{s} + F(s)\Big) \cdot S'(s) ds \quad \text{for all } t \geq \alpha.$$

Then, by arguing as in Section 2, one deduces the "concrete" relation

$$f \in L_{\Phi_0}(\mathbb{T}) \quad \text{if, and only if,} \quad M(f) \in L_{F_\alpha \cdot S}(\mathbb{T}),$$

for any $\alpha \geq \alpha_0$.

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