Calderón couples of Lorentz spaces

by

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Abstract

We present some sufficient conditions under which pairs of Banach function lattices are Calderón couples with special attention to the case of classical Lorentz spaces.

1. Introduction

Let us start by recalling some standard conventions of interpolation theory.

Suppose that $\bar{X} = (X_0, X_1)$, $\bar{Y} = (Y_0, Y_1)$ are two quasi–Banach couples (see [BK] and [BL]). We denote $\mathcal{L}(\bar{X};\bar{Y})$ the space of linear operators $T : X_0 + X_1 \rightarrow Y_0 + Y_1$ such that $T(X_j) \subset Y_j$ ($j = 0, 1$) and $\|T\|_{\bar{X},\bar{Y}} = \max(\|T\|_{X_0,Y_0}, \|T\|_{X_1,Y_1}) < \infty$. When $\bar{Y} = \bar{X}$, $\mathcal{L}(\bar{X}) = \mathcal{L}(\bar{X};\bar{Y})$. An interpolation pair $(X, Y)$ relative to $(\bar{X}, \bar{Y})$ is a pair of intermediate quasi–Banach spaces (i.e. $X_0 \cap X_1 \subset X \subset X_0 + X_1$ and $Y_0 \cap Y_1 \subset Y \subset Y_0 + Y_1$, with continuous embeddings) such that $T(X) \subset Y$ if $T \in \mathcal{L}(\bar{X};\bar{Y})$.

If $K(x, t; \bar{X})$ is the Peetre $K$–functional, i.e.

$$K(x, t; \bar{X}) = \inf \{ \|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1 \} \quad (x \in X_0 + X_1, \ t > 0),$$

the pair $(\bar{X}, \bar{Y})$ of quasi–Banach couples is called a relative Calderón couple when the following property holds:

For any pair $(X, Y)$ of interpolation spaces relative to $(\bar{X}, \bar{Y})$, there exists a constant $C_{X,Y} > 0$ such that, if

$$K(x, t; \bar{X}) \leq K(y, t; \bar{Y}) \quad (t > 0)$$

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and \( y \in Y \), then \( x \in X \) with \( \| x \|_X \leq C_{X,Y} \| y \|_Y \).

To check this property it is enough to prove that there exists an operator \( S \in \mathcal{L}(\overline{Y}; \overline{X}) \) and a constant \( C > 0 \) such that \( \| S \|_{\overline{Y}, \overline{X}} \leq C \) and \( Sy = x \).

The importance of these couples arises from the fact that, for them, all possible interpolation results can be described in terms of \( K \)–functional inequalities.

In the case \( \overline{Y} = \overline{X} \), if the above property holds when \( X = Y \), \( \overline{X} \) is said to be a Calderón couple. For any Calderón couple \( \overline{X} \) of Banach spaces, as a consequence of the \( K \)–divisibility theorem of Brudny and Krugljak, any interpolation Banach space \( X \) can be described by the so–called \( K \)–method, i.e., \( X = K_X(\overline{X}) \), defined by the norm

\[
\| x \|_X = \| K(x, \cdot; \overline{X}) \|_\Phi \quad (x \in X)
\]

for a suitable Banach lattice \( \Phi \) of measurable functions on \( \mathbb{R}^+ \) (see [BK]).

Since the result of Calderón [Ca] and Mityagin [Mi] which states that \( (L_1, L_\infty) \) is a Calderón couple, a considerable amount of work has been devoted to describe the pairs of Banach couples with this property. It is well known that any pair of \( L_p \)–spaces is a Calderón couple, and so are the couples \( (L_{p_0}(\omega_0), L_{p_1}(\omega_1)) \) of weighted \( L_p \)–spaces (cf. [Cw1] and [Sp]), and that a couple of interpolated spaces \( (\overline{X}_{p_0, \vartheta_0}, \overline{X}_{p_1, \vartheta_1}) \) \( (0 < \vartheta_0, \vartheta_1 < 1 \) and \( 0 < p_0, p_1 \leq \infty \)) is always a Calderón couple (cf. [Cw2]).

We are specially concerned with Lorentz spaces

\[
\Lambda^p_\omega(u) = \{ f \in L_0; \| f \|_{\Lambda^p_\omega(u)} = \| f^*_u \|_{L_p(\omega)} < \infty \}, \tag{1}
\]

where \( u \) and \( w \) are two weight functions on \( \mathbb{R}^+ \), \( f^*_u(s) = \inf \{ \lambda > 0; \int_{\{|f| > \lambda\}} u(t) \, dt \leq s \} \), and we write \( f^* = f^*_u \) and \( \Lambda^p(\omega) = \Lambda^p_\omega(u) \) if \( u(t) \equiv 1 \).

It has been shown that, under certain conditions, pairs of Lorentz spaces are Calderón couples. See Cwikel [Cw2], Merucci [Me], Cwikel and Nilson [CN], and Kalton, who in the paper [Ka] presents a very general criterion to determine the symmetric Calderón couples, which applies to many special cases. More recently, Brudnyi and Shteinberg have found a sufficient condition for a pair \( (\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1)) \) to be Calderón couple (see [BrS; Theorem 1.5]).

In general, \( \Lambda^p_\omega(u) \) need not be normable. It is known (cf. [Sw] and [CGS]) that \( \Lambda^p(\omega) \) is a Banach space (for a norm equivalent to \( \| \cdot \|_{\Lambda^p(\omega)} \)) if and only if \( w \) satisfies

\[
t^p \int_t^\infty \frac{\omega(x)}{x^p} \, dx \leq C \int_0^t \omega(x) \, dx \quad \text{when} \quad 1 < p < \infty
\]

and, in the case \( p = 1 \), the quasi–concavity condition for \( \int_0^t \omega(x) \, dx \)

\[
\frac{1}{t} \int_0^t \omega(x) \, dx \leq C \frac{1}{s} \int_0^s \omega(x) \, dx \quad \text{if} \quad t \geq s.
\]
We shall write \( \omega \in B_p \) if \( \Lambda^p(\omega) \) has this property (in the case \( p = 1 \), it is usually written \( \omega \in B_{1,\infty} \)).

If we only require quasi–normability, \( \| \cdot \|_{A^*_p(\omega)} \) is a quasi–norm if and only if \( W(x) := \int_0^x \omega(t) \, dt \) has the doubling property \( W(2x) \leq CW(x) \) (see [CS1]).

The above \( B_p \)–condition is related to the boundedness of the Hardy operator

\[
P f(t) = \frac{1}{t} \int_0^t f(s) \, ds
\]
on decreasing functions (cf. [AM]). The Hardy–conjugate operator

\[
Q f(t) = \int_t^\infty f(s) \frac{ds}{s}
\]
will also play an important role. It is bounded in \( L_p(\omega) \) when \( w \) satisfies the \( A^*_p \)–condition:

\[
\left( \int_0^t \omega(x) \, dx \right)^{1/p} \left( \int_t^\infty \frac{\omega(s)^{-p'/p}}{s^{p'}} \, ds \right)^{1/p'} \leq C \quad \text{when} \quad 1 < p < \infty
\]
and

\[
\frac{1}{t} \int_0^t \omega(x) \, dx \leq C \omega(t) \quad \text{when} \quad p = 1.
\]

The plan of the paper is as follows. In section 2 we study conditions on a Calderón couple \( \bar{X} \) of general Banach lattices under which the associated symmetric pair of spaces \( \bar{X}^s \) is also a Calderón couple. We observe that interpolation of operators restricted to decreasing functions for the so–called pairs with Marcinkiewicz cones of decreasing functions in [Sa] or Marcinkiewicz couples in [CM1] appear naturally.

Section 3 is devoted to the classical Lorentz spaces and their two–weighted version. If \( \omega_j \in B_p \cap A^*_p \) \((j = 0, 1)\), our main results state that \( (\Lambda^p(\omega_0), \Lambda^p(\omega_1)) \), and also \( (\Lambda^p_0(\omega_0), \Lambda^p_1(\omega_1)) \) when \( \int_0^\infty u(s) \, ds = \infty \), is a Calderón couple. This includes the cases considered in [Ka], where the weights are assumed to fulfill some growth conditions. It would also be interesting to relate our results, which are stated directly in terms of the weights, with those of [BrS], in terms of a factorization property of the fundamental functions.

As usual, \( A \simeq B \) means that \( A \leq C_1 B \) and \( B \leq C_2 A \) for some constants \( C_1, C_2 > 0 \). If the quasi-Banach space \( X \) is given, we will set \( \|g\|_X = \infty \) if \( g \notin X \).

### 2. Banach lattices

Throughout this paper \( L_0 \) represents the vector lattice of all real measurable functions (more precisely, of classes of equivalent functions) on \( \mathbb{R}^+ = (0, \infty) \), which is endowed
with Lebesgue measure. A decreasing function will be a non–increasing and non–negative function on \( \mathbb{R}^+ \), and, for any function space \( X \subset L_0 \) on \( \mathbb{R}^+ \), the set of all decreasing functions of \( X \) will be denoted \( X^d \).

A Banach lattice, \( X \), will be a Banach space which is a linear subspace of \( L_0 \) with the following properties:

**Ideal property.**- If \( f, g \in L_0 \) are such that \( |f| \leq |g| \) (the usual ordering in \( L_0 \)), then \( \|f\|_X \leq \|g\|_X \).

**Fatou property.**- If \( 0 \leq f_n \uparrow f \) a.e. with \( f_n \in X \) and \( \sup_n \|f_n\|_X < \infty \), then \( \lim_n \|f_n\|_X = \|f\|_X \).

**Köthe condition.** \( \chi_A \in X \) for every \( A \subset \mathbb{R}^+ \) of finite Lebesgue measure.

We will also consider cases where \( \|\cdot\|_X \) is a quasi–norm instead of a norm, and then we say that \( X \) is a quasi–Banach lattice if \( \|f\|_X \leq C\|g\|_X \) for some constant \( C > 0 \) whenever \( |f| \leq |g| \).

We say that \( X \) is symmetric if \( \|f\|_X \simeq \|g\|_X \) for every pair of equimeasurable functions.

We associate to any quasi–Banach lattice \( X \) the set

\[ X^* = \{ f \in L_0; \|f\|_{X^*} < \infty \}, \text{ with } \|f\|_{X^*} = \|f^*\|_X. \]

If \( X = L_p(\omega) \), with \( \|f\|_X = \|f\|_{L_p(\omega(t)dt)} \), then \( X^* = \Lambda^p(\omega) \).

If \( X^* \) is a quasi–normed space (for a quasi–norm equivalent to \( \|\cdot\|_{X^*} \)), \( (X^*, \|\cdot\|_{X^*}) \) is a quasi–Banach symmetric lattice and it follows from the results in [HM] that the operator

\[ D_2f(t) = f(t/2) \]

is bounded. We have the following converse:

**Lemma 1.** Let \( X \) be a quasi–Banach lattice such that \( D_2 \) is bounded when restricted to decreasing functions of \( X \). Then

(a) \( (X^*, \|\cdot\|_{X^*}) \) is a quasi–Banach lattice, and

(b) All dilation operators, \( D_a f(t) = f(t/a) \) (\( a > 0 \)), are bounded on \( X^* \).

**Proof:** Obviously, \( \|f + g\|_{X^*} \leq \|D_2f^* + D_2g^*\|_X \leq C(\|f\|_{X^*} + \|g\|_{X^*}). \) Since \( |f| \leq |g| \) implies \( f^* \leq g^* \), \( X^* \) has the ideal property, and it has also the Fatou property and satisfies the Köthe condition.

The completeness is a consequence of Fatou property and then (b) follows from [HM].
An operator of Hardy–conjugate type on $X$ will be a linear operator

$$T : \mathcal{D}(T) \subset L_0 \longrightarrow L_0$$

such that $X, X^* \subset \mathcal{D}(T)$ and with the following properties:
(i) $T : X \longrightarrow X$, bounded,
(ii) $T : X^* \longrightarrow X^*$, bounded,
(iii) $Tf$ is decreasing if $f \geq 0$,
(iv) $f \leq Tf$ if $f \in \mathcal{D}(T)$ is decreasing, and
(v) $|Tf| \leq T(|f|)$ if $f, |f| \in \mathcal{D}(T)$.

**Lemma 2.** Let $X$ be a quasi–Banach lattice with an operator $T$ of Hardy–conjugate type. Then,
(a) $D_2$ is bounded on decreasing functions, and $X^*$ is a quasi–Banach lattice.
(b) $T \in \mathcal{L}(X; X^*)$ and $T \in \mathcal{L}(X^*; X)$.

**Proof:** To prove (a), consider a decreasing simple function

$$s = \sum_{j=1}^{N} \alpha_j \chi_{E_j} \quad (E_i \cap E_j = \emptyset \text{ if } i \neq j).$$

Let $E_j = E_j^1 \cup E_j^2$ with $|E_j| = 2|E_j^1|$, and

$$s^i = \sum_{j=1}^{N} \alpha_j \chi_{E_j^i} \quad (i = 1, 2).$$

Then $s^* = (D_2 s^i)^*$, with $D_2 s \leq TD_2 s$, and

$$\|D_2 s\|_X \leq \|TD_2(s^1 + s^2)\|_X \leq \|TD_2 s^1\|_{X^*} + \|TD_2 s^2\|_{X^*} \leq C\|s\|_{X^*} = C\|s\|_X.$$ 

By Lemma 1, $X^*$ is a quasi–Banach lattice.

To prove (b), notice that $|Tf|^* \leq (T|f|)^* = T(|f|)$ and it follows that $T : X \longrightarrow X^*$ and $T : X^* \longrightarrow X$, since

$$\|Tf\|_{X^*} = |||Tf||^*|_X \leq \|T(|f|)\|_X \leq C\|f\|_X$$

and similarly

$$\|Tf\|_X \leq |||T(|f|)||_{X^*} = \|T(|f|)\|_{X^*} \leq C\|f\|_{X^*}. \quad \square$$

**Example 1.** Let $X$ be a Banach lattice such that $X^*$ is normable. If $Q : X \longrightarrow X$ is bounded, then $D_2 Q$ is of Hardy–conjugate type on $X$. 

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Proof: To check property (iv), observe that, if \( f \) is decreasing,
\[
\log 2 f(2t) \leq \int_t^{2t} f(s) \frac{ds}{s} \leq \int_t^{\infty} f(s) \frac{ds}{s}.
\]

Now we only need to prove (ii) or, since \( D_2 \) is bounded on \( X^s \) (a symmetric lattice), that \( Q : X^s \to X^s \) is bounded.

But, when restricted to decreasing functions, \( Q \) is bounded (it is bounded on \( X \)) and, for the Hardy operator
\[
P f(t) = \frac{1}{t} \int_0^t f(s) ds,
\]
we have \( PQ(|f|) \leq PQ(f^*) \), thus \( K(Q(|f|), t; L_1, L_\infty) \leq K(Q(f^*), t; L_1, L_\infty) \). Since \( X^s \) is a symmetric Banach lattice, it is an interpolation space for the Calderón couple \((L_1, L_\infty)\) (see [BS]) and then
\[
\|Q(f)\|_{X^s} \leq \|Q(|f|)\|_{X^s} \leq \|Q(f^*)\|_{X^s} \leq C\|f^*\|_{X^s} = C\|f\|_{X^s}
\]
as announced. \( \square \)

Similar examples of Hardy–conjugate type can be constructed by considering \( D_2Q_\alpha \) with
\[
Q_\alpha f(t) = \frac{1}{t^\alpha} \int_t^{\infty} s^\alpha f(s) \frac{ds}{s}
\]
for any \( \alpha \in [0, 1] \), instead of \( D_2Q \).

If \( \tilde{X} \) is a couple of quasi–Banach lattices with an operator \( T \) with properties (i), (iii) and (iv) of Hardy–conjugate type operators (on both \( X_0 \) and \( X_1 \)), then, as observed in [CM1], \( \tilde{X} \) is a Marcinkiewicz couple, which means that
\[
K(f, t) \simeq K^d(f, t) \quad (t > 0, \ f \in (X_0 + X_1)^d),
\]
where \( K(f, t) = K(f, t; \tilde{X}) \) is Peetre’s \( K \)–functional and
\[
K^d(f, t) = K(f, t, \tilde{X}^d) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1}; \ f = f_0 + f_1, \ f_j \in X_j^d\},
\]
the \( K \)–functional restricted to decreasing functions.

Theorem 1. Let \( \tilde{X} \) be a pair of Banach lattices with an operator \( T \in \mathcal{L}(\tilde{X}) \) of Hardy–conjugate type. If \( \tilde{X} \) is a Calderón couple, then \( \tilde{X}^s = (X_0^s, X_1^s) \) is also a Calderón couple (of quasi–Banach lattices).
**Proof:** We claim that, if $X$ is an interpolation space for $\tilde{X}^*$, it follows from the existence of $T \in \mathcal{L} (\tilde{X})$ of Hardy–conjugate type that $X$ is symmetric.

To prove this claim, assume that $f^* = g^*$. Then (see [KPS]), for every $\varepsilon > 0$, we can find a measure–preserving mapping $\omega$ from $(0, \infty)$ into itself and $\beta \in L_0$ such that $|\beta| \leq 1$ and

$$\|g - \beta \cdot (f \circ \omega)\|_{L_1 \cap L_\infty} \leq \varepsilon.$$  

The operator $T \omega f = f \circ \omega$ is bounded on $\tilde{X}^*$, and, by interpolation, it is also bounded on $X$. Hence

$$\|g\|_X \leq c(\|g - \beta \cdot (f \circ \omega)\|_X + \|\beta T \omega f\|_X) \leq C(\|g - \beta \cdot (f \circ \omega)\|_X + \|f\|_X). \quad (2)$$

We only need to prove that $L_1 \cap L_\infty$ is continuously embedded in $X_k^\varepsilon$ ($k = 0, 1$), since then $L_1 \cap L_\infty \subset X$ and $\|g\|_X \leq C(\|f\|_X + \varepsilon)$. The same argument also proves that $\|g\|_X \leq C\|f\|_X$.

Following for instance [KPS; Lemma II.4.1], for any decreasing $f \in L_1 \cap L_\infty$ we consider

$$\sum_{j=0}^\infty f(j+1)\chi_{[j,j+1)} \leq \sum_{j=0}^\infty f(j)\chi_{[j,j+1)},$$

where $\sum_{j=0}^\infty f(j+1) \leq \|f\|_1 < \infty$.

The partial sums $s_N = \sum_{j=0}^N f(j)\chi_{[j,j+1)}$ satisfy

$$(s_{N+p} - s_N)^* = \sum_{j=0}^{p-1} f(j+N+1)\chi_{[j,j+1)}$$

$$\leq T(\sum_{j=0}^{p-1} f(j+N+1)\chi_{[j,j+1)}) = \sum_{j=0}^{p-1} f(j+N+1)T(\chi_{[j,j+1)})$$

and, since for decreasing functions $\|f + g\|_{X_k^\varepsilon} \leq \|f\|_{X_k^\varepsilon} + \|g\|_{X_k^\varepsilon}$, if we denote $C = \|T\|_{X_k^\varepsilon, X_k^\varepsilon} \|\chi_{[0,1]}\|_{X_k}$,

$$\|s_{N+p} - s_N\|_{X_k^\varepsilon} \leq C \sum_{j=0}^{p-1} f(j+N+1) \to 0$$

as $N \uparrow \infty$. Thus $s_N \uparrow \tilde{f}(t) := \sum_{j=0}^\infty f(j)\chi_{[j,j+1)}$ and then $\|\tilde{f} - s_N\|_{X_k^\varepsilon} \to 0$ with

$$\|f\|_{X_k^\varepsilon} \leq \|\tilde{f}\|_{X_k^\varepsilon} \leq C(f(0) + \sum_{j=0}^\infty f(j+1)) \leq 2C\|f\|_{L_1 \cap L_\infty}.$$  

This ends the proof of the claim.

To prove the theorem, assume that $f, g$ are two functions in $X_0^* + X_1^*$ such that

$$K(f,t;\tilde{X}^*) \leq K(g,t;\tilde{X}^*). \quad (3)$$
We shall see that \( \|f\|_X \leq C\|g\|_X \) if \( X \) is an interpolation space for \( \bar{X}^s \).

Since \( X^*_0 \) and \( X^*_1 \) are symmetric, condition (??) is equivalent to

\[
K(f^*, t; \bar{X}^*) \leq CK(g^*, t; \bar{X}^*).
\]

If we prove that there exists an operator \( H \in \mathcal{L}(\bar{X}^s) \) such that \( f^* = Hg^* \), then

\[
\|f^*\|_X \leq \|H\|\|g^*\|_X.
\]

By our claim, \( X \) is symmetric, thus

\[
\|f\|_X \simeq \|f^*\|_X \leq \|H\|\|g^*\|_X \simeq \|H\|\|g\|_X.
\]

Hence, we can assume that \( f \) and \( g \) are decreasing. Then \( g \leq Tg \) and

\[
K(f, t; \bar{X}^*) \leq K(Tg, t; \bar{X}^*),
\]

and, since \( \bar{X}^s \) is a Marcinkiewicz couple (cf. [CM1]),

\[
K(\cdot, t; \bar{X}^*) \simeq K^d(\cdot, t; \bar{X}^*) = K^d(\cdot, t; \bar{X}).
\]

It follows from the properties of Hardy–conjugate type operators that, on decreasing functions, \( K^d(\cdot, t; \bar{X}) \simeq K(\cdot, t; \bar{X}) \), hence

\[
K(f, t; \bar{X}) \leq CK(Tg, t; \bar{X}). \tag{4}
\]

But \( \bar{X} \) is assumed to be a Calderón couple and there exists \( S \in \mathcal{L}(\bar{X}) \) such that \( f = STg \). Again, \( f \) being decreasing, \( f \leq Tg \) and, by Lemma 2, if we define

\[
Hh = \frac{fTSTh}{TSTg},
\]

we obtain an operator \( H \in \mathcal{L}(\bar{X}^s) \) such that \( Hg = f \).

**Corollary 1.** Let \( \bar{X} \) be a couple of Banach lattices such that \( X^*_0 \) and \( X^*_1 \) are normable and \( Q \in \mathcal{L}(\bar{X}) \). Then, if \( \bar{X} \) is a Calderón couple, so is also \( \bar{X}^s \).

**Remark 1.** For any Calderón couple \( \bar{X} \) of Banach lattices with an operator \( T \in \mathcal{L}(\bar{X}) \) of Hardy–conjugate type, \((\bar{X}^s, \bar{X})\) and \((\bar{X}, \bar{X}^s)\) are two relative Calderón couples.

As an application, despite that in Theorem 1 the spaces of the Calderón couple \( \bar{X}^s \) are quasi–normed, any interpolation space \( X \) of this couple is still described by the \( K \)–method, as in the case of couples of Banach spaces.
To see that \((X^*, \bar{X})\) is relative Calderón couple we can follow the proof of Theorem 1. If \(K(f, t; \bar{X}^*) \leq K(g, t; \bar{X})\), we can assume that \(f\) is decreasing and \(K(f, t; \bar{X}) \leq K(g, t; \bar{X})\). Since \(\bar{X}\) is a Calderón couple, \(f = Sg\) and we define

\[
Hh = \frac{fTSh}{TSG}.
\]

The same argument shows that \((\bar{X}, \bar{X}^*)\) is a relative Calderón couple.

Now, if \((Z_0, Z_1)\) is a relative interpolation pair for \((\bar{X}^*, \bar{X})\), since \(\bar{X}\) is a Banach couple, there exists a parameter \(\Phi\) such that \(Z_1 = K_\Phi(\bar{X})\).

By interpolation, \(T : Z_0 \rightarrow K_\Phi(\bar{X})\) and, if \(f \in Z_0^d, f \leq Tf \in K_\Phi(\bar{X})\). Thus, \(Z_0^d \hookrightarrow K_\Phi(\bar{X})^d\). Similarly, \(T : K_\Phi(\bar{X}) \rightarrow Z_0\) and \(K_\Phi(\bar{X})^d \hookrightarrow Z_0^d\). So, \(K_\Phi(\bar{X})^d = Z_0^d\) and \(K_\Phi(\bar{X}^*) = K_\Phi(\bar{X})^d\), since \(K(\cdot, t; \bar{X}^*) \simeq K(\cdot, t; \bar{X})\) on decreasing functions as in the proof of Theorem 1.

But \(K_\Phi(\bar{X}^*)\) and \(Z_0\) are symmetric, they are determined by the corresponding cone of decreasing functions and it follows that \(K_\Phi(\bar{X}^*) = Z_0\)

**Theorem 2.** Let \(X\) be a Banach lattice such that \(X^*\) is normable and \(Q \in \mathcal{L}(X)\). If \(Y\) is a symmetric Banach lattice such that \((X,Y)\) is a Calderón couple, then \((X^*, Y)\) is also a Calderón couple.

**Proof:** Let us prove that \((X,Y)\) is a Marcinkiewicz couple.

Denote \(Q_0f(t) = \int_2^{2t} f(s) \, ds/s\). It follows from \(|Q_0f| \leq Q(|f|)\) that \(Q_0(L_1) \subset L_1\), and also \(Q_0(L_\infty) \subset L_\infty\), hence we obtain \(Q_0(Y) \subset Y\) by interpolation.

To prove that \(K_d(f, t) \simeq K(f, t)\) for decreasing functions, let \(f = f_0 + f_1\) be such that \(\|f_0\|_X + t\|f_1\|_Y \leq 2K(f, t)\), with \(f_j \geq 0 (j = 0, 1)\). Then, since \(f\) is decreasing,

\[
f(2t) \log 2 \leq Q_0(f_0) + Q_0(f_1) \leq Q(f_0) + Q(f_1)
\]

and

\[
f(2t) = f^*(2t) \leq C(D_2Q(f_0) + D_2(Q_0f_1)^*),
\]

hence \(f \leq C(D_4Q(f_0) + D_4(Q_0f_1)^*)\). But, by the decomposition properties of decreasing functions (cf. \([CM1]\)), there exist two decreasing functions \(g_0 \leq CD_4Q(f_0)\) and \(g_1 \leq CD_4(Q_0f_1)^*\) such that \(f = g_0 + g_1\).

Thus

\[
K^d(f, t) \leq \|g_0\|_X + t\|g_1\|_Y \leq C(\|D_4Q(f_0)\|_X + t\|D_4(Q_0f_1)^*\|_Y)
\]

\[
\leq C'(\|f_0\|_X + t\|f_1\|_Y) \leq 2C'K(f, t)
\]

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and \((X, Y)\) is a Marcinkiewicz couple.

Now we can continue as in the proof of Theorem 1, with the Hardy–conjugate type operator \(D_2Q_0\) instead of \(D_2Q\) of Example 1. □

**Remark 2.** In particular, if \(X\) is a Banach lattice such that \(X^*\) is normable and \(Q \in \mathcal{L}(X)\), then, if \((X, L_\infty)\) is a Calderón couple, \((X^*, L_\infty)\) is also a Calderón couple.

Theorem 1 has the following extension for relative Calderón couples.

**Theorem 3.** (a) Let \(\tilde{X}\) and \(\tilde{Y}\) be two pairs of Banach lattices, every one with a Hardy–conjugate type operator. If \((\tilde{X}, \tilde{Y})\) is a relative Calderón couple, then both \((\tilde{X}^*, \tilde{Y}^*)\), \((\tilde{X}^*, \tilde{Y})\) and \((\tilde{X}, \tilde{Y}^*)\) are also relative Calderón couples.

(b) Let \(\tilde{X}\) be a Marcinkiewicz couple of Banach lattices. If \(Q \in \mathcal{L}(\tilde{X}^*)\) (or if \(P \in \mathcal{L}(\tilde{X}^*)\)) and \(\tilde{X}\) is a Calderón couple, then \((\tilde{X}^*, \tilde{X})\) is a relative Calderón couple.

(c) If \(Q \in \mathcal{L}(\tilde{X}), L_1 \cap L_\infty \subset X_i^* \,(i = 0, 1)\) and \(\tilde{X}\) is a Calderón couple, then \((\tilde{X}, \tilde{X}^*)\) is a relative Calderón couple.

**Proof:** (a) follows as Theorem 1. For example, in the case of \((\tilde{X}^*, \tilde{Y}^*)\), let

\[
K(f, t; \tilde{X}^*) \leq K(g, t; \tilde{Y}^*),
\]

where we can assume that \(f\) and \(g\) are decreasing. If \(T_1\) and \(T_2\) are the Hardy–conjugate type operators for \(\tilde{X}\) and \(\tilde{Y}\), respectively, it follows from \(K(f, t; \tilde{X}^*) \leq K(T_2g, t; \tilde{Y}^*)\) that \(K(f, t; \tilde{X}) \leq K(T_2g, t; \tilde{Y})\) and \(f = ST_2g\) for some \(S \in \mathcal{L}(\tilde{Y}; \tilde{X})\). Thus, \(f \leq T_1f = T_1ST_2g\) and for

\[
Hh = \frac{fT_1ST_2h}{T_1ST_2g}
\]

defines \(H \in \mathcal{L}(\tilde{Y}^*; \tilde{X}^*)\) such that \(Hg = f\).

To check (b), let

\[
Q_0f(t) := \int_t^{2t} f(s) \frac{ds}{s} = \int_t^\infty (f(s) - f(2s)) \frac{ds}{s}.
\]

As in the proof of Theorem 1, if \(Q \in \mathcal{L}(\tilde{X}^*)\), then \(L_1 \cap L_\infty \subset \tilde{X}^*_i \,(i = 0, 1)\). Moreover, \(Q_0 \in \mathcal{L}(\tilde{X}^*_i; \tilde{X})\), since

\[
\|Q_0f\|_X \leq \|Q(\|f(s) - f(2s)\|)\|_{\tilde{X}^*} \leq C\|f\|_{\tilde{X}^*}.
\]

If \(P \in \mathcal{L}(\tilde{X}^*_i)\), then \(\tilde{X}^*_0\) and \(\tilde{X}^*_1\) are normable. Write \(Q_0f = P(2f(2s) - f(s))\) and, since \(P(2f(2s) - f(s)) \leq P((2f(2s) - f(s))^*)\),

\[
\|Q_0f\|_X \leq \|P((2f(2s) - f(s))^*)\|_X = \|P((2f(2s) - f(s))^*)\|_{\tilde{X}^*} \leq C\|f\|_{\tilde{X}^*}
\]

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and also \( Q_0 \in \mathcal{L}(X^*; X) \).

Now (b) follows as Theorem 1, with \( D_2 Q_0 \) instead of \( T \).

In case (c), since \( Q \in \mathcal{L}(\tilde{X}) \), \( \tilde{X} \) is a Marcinkiewicz couple (see [CM1]), \( D_2 \) is bounded on decreasing functions and \( Q \in \mathcal{L}(X; X^*) \).

Let us close this section with some remarks concerning operators on a quasi–Banach lattice \( X \) which are bounded when restricted to decreasing functions, i.e., such that

\[
\|Tf\|_X \leq C\|f\|_X \quad (f \in X^d)
\]

for some constant \( C > 0 \). These operators have been considered recently by many authors and it is natural to ask whether \( T \) is bounded on decreasing functions of an interpolation space \( X \) of a given couple \( \tilde{X} \) of quasi–Banach lattices whenever \( T \) (defined on \( X_0 + X_1 \)) is bounded when restricted to decreasing functions of \( X_0 \) and of \( X_1 \).

**Proposition 1.** Let \( \tilde{X} \) and \( \tilde{Y} \) two couples of quasi–Banach lattices with the following properties:

(i) For the couple \( \tilde{X} \), \( K^d(f, t) \simeq K(f, t) \) on decreasing functions, and

(ii) \((\tilde{Y}, \tilde{X})\) is a relative Calderón couple.

If \( X \) and \( Y \) are relative interpolation spaces for \( \tilde{X} \) and \( \tilde{Y} \), and \( T : \tilde{X} \to \tilde{Y} \) is a bounded operator on decreasing functions, then \( T : X^d \to Y \) is also a bounded operator.

**Proof:** Let \( f \in X^d \). If we consider decompositions \( f = f_0 + f_1 \) with \( f_j \in X_j^d \) \((j = 0, 1)\), then

\[
\|Tf_0\|_{Y_0} + t\|Tf_1\|_{Y_1} \leq C(\|f_0\|_{X_0} + t\|f_1\|_{X_1}),
\]

hence

\[
K(Tf, t; \tilde{Y}) \leq C_1 \inf_{f=f_0+f_1} (\|f_0\|_{X_0} + t\|f_1\|_{X_1}) \leq C_1 K^d(f, t; \tilde{X}) \leq C_2 K(f, t; \tilde{X}).
\]

Since \((\tilde{X}, \tilde{Y})\) is a relative Calderón couple, \( \|Tf\|_Y \leq C_3\|f\|_X \). \( \square \)

**Remark 3.** It is easily seen that in Proposition 1 we cannot drop any of conditions (i) and (ii). For condition (i) consider the couples \((L_1, L_1(dt/t)), (L_1(dt/t), L_1)\) and \( T(t) = tf(t) \). Obviously \( T : L_1^d \to L_1(dt/t) \) and \( T : L_1(dt/t)^d = \{0\} \to L_1 \).

Then \((L_1, L_1(dt/t))_{1/2} = (L_1(dt/t), L_1)_{1/2} = L_1(dt/\sqrt{t}) \) and, for instance, if \( f(s) = \chi_{[0,1]}(s) + (1/s)\chi_{[1,\infty)}(s) \), we obtain \( f \in L_1(dt/\sqrt{t})^d \) but \( Tf \notin L_1(dt/\sqrt{t}) \).

Condition (ii) is needed if quasi–linear operators are permitted (see [Cw3]).
3. Lorentz spaces

As a direct application of Corollary 1 we have:

**Theorem 4.** Let $1 \leq p_j < \infty$. If $\omega_j \in B_{p_j} \cap A_{p_j}^*$ ($j = 0, 1$), then $(\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1))$ is a Calderón couple of Banach spaces.

Similarly, as an application of Theorem 3:

**Theorem 5.** (a) $1 \leq p_j \leq q_j < \infty$. If $\omega_j \in B_{p_j} \cap A_{p_j}^*$ and $v_j \in B_{q_j} \cap A_{q_j}^*$ ($j = 0, 1$), then $\left( (\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1)), (\Lambda^{q_0}(v_0), \Lambda^{q_1}(v_1)) \right)$ is a relative Calderón couple.

(b) Let $1 \leq p_0, p_1 < \infty$ and assume that $(L_{p_0}(\omega_0), L_{p_1}(\omega_1))$ is a Marcinkiewicz couple. Assume also that, for $j = 0, 1$,

(i) $\omega_j \in B_{p_j}$ and $PP \omega_j \leq CP \omega_j$

or

(ii) $\omega_j \in B_{p_j}$, if $1 < p_j < \infty$, and $Q \omega_j \leq CP \omega_j$ if $p_j = 1$.

Then $\left( (\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1)), (L_{p_0}(\omega_0), L_{p_1}(\omega_1)) \right)$ is a relative Calderón couple.

(c) Consider $\omega_j \in A_{p_j}^*$ such that $L_1 \cap L_\infty \hookrightarrow \Lambda^{p_j}(\omega_j)$ ($j = 0, 1$). Then $(L_{p_0}(\omega_0), L_{p_1}(\omega_1)), (\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1))$ is a relative Calderón couple.

**Proof:** For (b) we apply Theorem 3. In both cases the spaces $\Lambda^{p_j}(\omega_j)$ are normable, in case (i) we have $Q \in \mathcal{L}(\Lambda^{p_j}(\omega_j))$ (cf. [Ne]), and in case (ii) $P \in \mathcal{L}(\Lambda^{p_j}(\omega_j))$. $\square$

**Theorem 5.** Let $\omega_0$ and $\omega_1$ be two decreasing weights. If there exist two constants $r_0, r_1 > 1$ such that

$$\inf_{s > 0} \frac{\omega_j(r_j s)}{\omega_j(s)} > \frac{1}{r_j} \quad (j = 0, 1),$$

then $(\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1))$ is a Calderón couple of Banach spaces.

**Proof:** In [CM2] we have proved that, for a decreasing weight $\omega$ (which is always in $B_p$, for any $p \in [1, \infty)$), condition

$$\inf_{s > 0} \frac{\omega(rs)}{\omega(s)} > \frac{1}{r}$$

($r > 1$) is equivalent to $\omega \in A^*_p$ for one or all $p \in [1, \infty)$. $\square$

**Theorem 6.** Let $1 \leq p_j < \infty$ and $\omega_j \in B_{p_j}$ ($j = 0, 1$). If $u$ is a weight such that $\int_0^\infty u(s) ds = \infty$, then $\Lambda^u_\infty(\omega) := (\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1))$ is a Calderón couple if and only if $\Lambda^\infty(\omega) := (\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1))$ is also a Calderón couple.
Proof: Condition $\int_{0}^{\infty} u(s) \, ds = \infty$ (with $\omega_j \in B_{p_j}$) ensures that $\Lambda_{\omega_0}^{p_0}$ and $\Lambda_{\omega_1}^{p_1}$ are Banach spaces, since in this case, as proved in [CGS], the normability of $\Lambda_{\omega_j}^{p_j}$ is equivalent to that of $\Lambda^{p_j}(\omega_j)$.

We claim that

$$K(f, t; \Lambda_{\omega}^{p}(\bar{\omega})) = K(f_{\omega}^{*}, t; \Lambda_{\bar{\omega}}^{p}(\omega)).$$

To prove this claim, we observe that $\Lambda_{\omega_j}^{p_j}(\omega_j)$ is an interpolation space of $(L_1(u), L_\infty)$, since they are normable, and there is an operator $R_0 \in \mathcal{L}(\Lambda_{\omega}^{p}(\bar{\omega}); \Lambda_{\bar{\omega}}^{p}(\omega))$ of norm at most 1, such that $R_0 f = f_{\omega}^{*}$ (see [BS; page 117]). Hence

$$K(f_{\omega}^{*}, t; \Lambda_{\omega}^{p}(\bar{\omega})) = K(R_0 f, t; \Lambda_{\bar{\omega}}^{p}(\omega)) \leq K(f, t; \Lambda_{\omega}^{p}(\bar{\omega})).$$

Now let $\sigma$ be any measure–preserving mapping $\sigma$ between $(\mathbb{R}^+, dt)$ and $(\mathbb{R}^+, u(s) \, ds)$ and define $R_{\sigma} h = h \circ \sigma$. On $(\mathbb{R}^+, u(s) \, ds)$, the functions $f$ and $R_{\sigma} f_{\omega}^{*}$ are equimeasurable. Thus

$$K(f, t; \Lambda_{\omega}^{p}(\bar{\omega})) = K(R_{\sigma} f_{\omega}^{*}, t; \Lambda_{\omega}^{p}(\bar{\omega})) \leq K(f_{\omega}^{*}, t; \Lambda_{\bar{\omega}}^{p}(\omega)),
$$

since $R_{\sigma} \in \mathcal{L}(\Lambda_{\omega}^{p}(\bar{\omega}); \Lambda_{\omega}^{p}(\bar{\omega}))$. This ends the proof of the claim.

Assume that $\Lambda_{\omega}^{p}(\bar{\omega})$ is a Calderón couple and

$$K(f, t; \Lambda_{\omega}^{p}(\bar{\omega})) \leq K(g, t; \Lambda_{\omega}^{p}(\bar{\omega})).$$

Since this is equivalent to

$$K(f_{\omega}^{*}, t; \Lambda_{\omega}^{p}(\bar{\omega})) \leq K(g_{\omega}^{*}, t; \Lambda_{\omega}^{p}(\bar{\omega})),
$$

there exists $G \in \mathcal{L}(\Lambda_{\omega}^{p}(\bar{\omega}))$ such that $G g_{\omega}^{*} = f_{\omega}^{*}$. Define $T := R_{\sigma} G R_0$ with $R_0 g_{\omega} = g_{\omega}^{*}$. Then $T g = f_{\omega}^{*} \circ \sigma$ and, since any interpolation space $X$ of $\Lambda_{\omega}^{p}(\bar{\omega})$ is rearrangement invariant and $f$, $f_{\omega}^{*} \circ \sigma$ are equimeasurable,

$$\|f\|_x = \|f_{\omega}^{*} \circ \sigma\|_x = \|T g\|_x \leq C\|g\|_x.
$$

To prove the converse, let $\Lambda_{\omega}^{p}(\bar{\omega})$ be a Calderón couple and

$$K(f, t; \Lambda_{\omega}^{p}(\bar{\omega})) \leq K(g, t; \Lambda_{\omega}^{p}(\bar{\omega})),
$$

where we can assume that $f$ and $g$ are decreasing.

If $\sigma$ is a measure–preserving mapping between $(\mathbb{R}^+, u(s) \, ds)$ and $(\mathbb{R}^+, dt)$, $f_0 = f \circ \sigma$ and $g_0 = g \circ \sigma$, then

$$K(f_0, t; \Lambda_{\omega}^{p}(\bar{\omega})) \leq K(g_0, t; \Lambda_{\omega}^{p}(\bar{\omega})).
$$

and $f_0 = G g_0$ for some $G \in \mathcal{L}(\Lambda_{\omega}^{p}(\bar{\omega}))$. 

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We choose a second measure–preserving mapping $\alpha$, now between $(\mathbb{R}^+, dt)$ and $(\mathbb{R}^+, u(s)ds)$, and define $R_\sigma h = h \circ \alpha$ and $R_\alpha h = h \circ \sigma$ to obtain $Tg = f_0 \circ \alpha$ for $T = R_\sigma GR_\alpha$.

**Remark 4.** Let $1 \leq p < \infty$. Assume that $(L_p(\omega_0), L_p(\omega_1))$ is a Marcinkiewicz couple, i.e., $K(f, t) \simeq K^d(f, t)$ on decreasing functions. This happens (cf. [CM1]) if and only if
\[
\|\chi_{(0,t)} \min(\omega_0, s \omega_1)\|_{L_p} \simeq \min(\|\chi_{(0,t)} \omega_0\|_{L_p}, s \|\chi_{(0,t)} \omega_1\|_{L_p}).
\]

If $\omega_0, \omega_1 \in B_p$, then $\left((\Lambda^p(\omega_0), \Lambda^p(\omega_1)), (L_p(\omega_0), L_p(\omega_1))\right)$ is a relative Calderón couple.

**Remark 5.** When $\omega_0$ and $\omega_1$ are decreasing weights, if $(L_{p_0}(\omega_0), L_{p_1}(\omega_1))$ is a Marcinkiewicz couple ($1 \leq p_0, p_1 < \infty$), then $\left((\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1)), (L_{p_0}(\omega_0), L_{p_1}(\omega_1))\right)$ is a relative Calderón couple, since $\Lambda^{p_j}(\omega_j) \hookrightarrow L_{p_j}(\omega_j)$ ($j = 0, 1$).

**Remark 6.** In the case of increasing weights, $\omega_0$ and $\omega_1$, if $L_1 \cap L_\infty \hookrightarrow \Lambda_{p_j}(\omega_j)$ and $0 < p_0, p_1 < \infty$, $\left((L_{p_0}(\omega_0), L_{p_1}(\omega_1)), (\Lambda^{p_0}(\omega_0), \Lambda^{p_1}(\omega_1))\right)$ is always a relative Calderón couple, since now $(L_{p_0}(\omega_0), L_{p_1}(\omega_1))$ is a Marcinkiewicz couple and $L_{p_j}(\omega_j) \hookrightarrow \Lambda^{p_j}(\omega_j)$ (see [CM3]).

**References**


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