ON THE RELATION BETWEEN RIESZ AND STRICTLY SINGULAR POSITIVE OPERATORS

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Abstract. The domination problem for Riesz operators is revisited with a particular emphasis on the relations with strictly singular operators.

1. Introduction

A classical question concerning the spectral theory of positive operators on Banach lattices is the domination problem for Riesz operators. Recall that an operator T on a Banach space is called Riesz if its essential spectrum (the spectrum in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$) reduces to $\{0\}$. Loosely speaking, the spectral theory for Riesz operators resembles that of compact operators. The fundamental problem under study is the following:

Problem 1.1. *Given a Banach lattice E and $0 \leq S \leq T : E \to E$ such that T is a Riesz operator, must S be also Riesz?*

A natural approach to this problem is through a desirable monotonicity property of the essential spectral radius r_{ess} (i.e. the spectral radius in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$):

Problem 1.2. *Suppose $0 \leq S \leq T : E \to E$, is it necessarilly $r_{ess}(S) \leq r_{ess}(T)$?*

These problems have been deeply studied by B. de Pagter and A. Schep in [19], and by J. Martínez and J. M. Mazón in [16] (see also [13, 20]). Problem 1.2 has in general a negative answer, as exhibited in [19, Example 3.7]. As far as we know, the answer to Problem 1.1 in full generality remains open.

Nevertheless, in some cases the essential spectral radius is monotonous. This is the case, for instance when E is a space of the form $L_1(\mu)$ or $C(K)$ (or more generally an AM-space) [16]. A preliminar version of this fact for a certain type of $L_1(\mu)$ spaces was also given by L. Weis and M. Wolff [24]. An important contribution to this question was made in [19, Theorem 3.2], where it is shown that $r_{ess}(S) \leq r_{ess}(T)$ provided $0 \leq S \leq T$ and S is AM-compact (that is $S[0, x]$ is relatively compact for every $x \in E_+$). In particular, this applies to operators on ℓ_p for $1 \leq p < \infty$, or c_0, and more generally, any order continuous Banach lattice endowed with the order given by an unconditional basis. It can also be applied to integral operators, thus completing some results for operators on rearrangement invariant spaces due to V. Caselles [5].

Our aim in this note is to explore the connection between Riesz and strictly singular operators (those not invertible on any infinite dimensional subspace), and benefit from the domination properties of the latter given in [8].

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2. Preliminaries

Let us recall some basic preliminaries on Banach lattices, positive operators and spectral theory. The reader is referred to the monographs [2] and [17] for further information about this framework.

A Banach lattice is a Banach space E endowed with a norm $\|\cdot\|$, and a partial order $\leq$ such that: if $x \leq y$, then $x+z \leq y+z$, for every $x, y, z \in E$; $\lambda x \geq 0$, for every $x \geq 0$ in E and every real number $\lambda \geq 0$; for every $x, y \in E$ there exist the least upper bound and the greatest lower bound of x, y in E; these are respectively denoted $x \vee y$ and $x \wedge y$; if we denote $|x| = x \vee (-x)$, it follows that $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$. A large class of Banach spaces arising in classical analysis are Banach lattices. For instance, this is the case for $L_p(\mu)$ spaces ($1 \leq p \leq \infty$), or more generally, Lorentz and Orlicz spaces, $C(K)$ spaces, Banach spaces with an unconditional basis...

A Banach lattice will be called Dedekind complete when every order bounded set has a supremum. A Banach lattice E has property (P) if there is a positive contractive projection $P : E'' \to E$. Note that a Banach lattice with property (P) is always Dedekind complete. Finally, recall that a Banach lattice is order continuous if every downwards directed set $x_\alpha \downarrow 0$ satisfies that $\inf_\alpha \|x_\alpha\| = 0$.

An operator between Banach lattices $T : E \to F$ is positive when it maps positive elements to positive elements. The linear space generated by positive operators will be denoted by $\mathcal{L}^r(E, F)$, and its elements will be referred to as regular operators (as usual we will denote $\mathcal{L}^r(E) = \mathcal{L}^r(E, E)$). The space $\mathcal{L}^r(E)$ endowed with the r-norm

$$\|T\|_r = \inf\{\|S\| : S \in \mathcal{L}(E), S \geq 0, |Tz| \leq S|z| \, \forall z \in E\}$$

is a Banach algebra (containing the unit of $\mathcal{L}(E)$), and when E is Dedekind complete it becomes a Banach lattice satisfying

$$\|T\|_r = \||T|\|.$$

It is clear that for $T \in \mathcal{L}^r(E)$, we have $\|T\| \leq \|T\|_r$. Hence, we have the continuous inclusion $\mathcal{L}^r(E) \hookrightarrow \mathcal{L}(E)$.

A general question concerning positive operators is the so-called domination problem. This consists in finding conditions so that whenever $0 \leq S \leq T : E \to F$ and T has certain property $\mathcal{P}$, then S also has property $\mathcal{P}$. This kind of problems have been studied for many different properties, such as several types of compactness. See [2, Chapter 5] and [9] for a survey on these problems.

Our interest in this note is to study the domination problem for Riesz operators. To introduce properly this class and its properties let us recall a few facts first (see [1, Chapter 7]). Given a Banach space X, the class of compact operators $\mathcal{K}(X)$ (i.e. those mapping the unit ball to a relatively compact set) is a closed ideal of the Banach algebra $\mathcal{L}(X)$. Therefore, we can consider the quotient algebra

$$\mathcal{C}(X) = \mathcal{L}(X)/\mathcal{K}(X)$$

which is usually called the Calkin algebra of X. Let us consider the natural quotient map

$$\pi : \mathcal{L}(X) \to \mathcal{C}(X)$$

which is in fact an algebra homomorphism.

The essential spectrum of an operator $T \in \mathcal{L}(X)$ is the spectrum of $\pi(T)$ in the Calkin algebra $\mathcal{C}(X)$:

$$\sigma_{ess}(T) = \{\lambda \in \mathbb{C} : \lambda I - \pi(T) \text{ is not invertible in } \mathcal{C}(X)\}.$$

Recall that an operator between Banach spaces $T : X \to Y$ is Fredholm when the dimension of the Kernel and the codimension of the range are both finite. A theorem of F. V. Atkinson on Fredholm operators (cf. [1, Theorem 4.46]) can be used to characterize the essential spectrum:

$$\sigma_{ess}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Fredholm}\}.$$

An operator $T \in \mathcal{L}(X)$ is called *Riesz* (or essentially quasi-nilpotent) when $\sigma_{ess}(T) = \{0\}$. This means that $I - \lambda T$ is a Fredholm operator for every scalar λ.

The essential spectral radius is defined by $r_{ess}(T) = r(\pi(T))$ and can be computed as

$$\begin{aligned} r_{ess}(T) &= \max\{|\lambda| : \lambda \in \sigma_{ess}(T)\} = \lim_{n\to\infty} \|\pi(T^n)\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \big(\inf_{K\in\mathcal{K}(X)} \|T^n - K\|\big)^{\frac{1}{n}}. \end{aligned}$$

Note that an operator $T \in \mathcal{L}(X)$ is Riesz if and only if $r_{ess}(T) = 0$.

It can also be seen (cf. [1, §7.5]) that $T \in \mathcal{L}(X)$ is a Riesz operator if and only if every $\lambda \in \sigma(T)\backslash\{0\}$ is an isolated point of $\sigma(T)$ and its spectral projection $P_\lambda(T)$ has finite rank. In particular, this implies that the spectrum of a Riesz operator is either finite or countable, its only accumulation point can be 0, and every non-zero point in the spectrum is an eigenvalue of T. Moreover, $T \in \mathcal{L}(X)$ is a Riesz operator if and only if T^n is Riesz for some $n \in \mathbb{N}$.

3. Riesz operators and strict singularity

An operator between Banach spaces is strictly singular if it is not invertible on any infinite dimensional closed subspace. This class of operators, which contains the compact operators, was introduced by T. Kato in [12] concerning perturbation theory. It turns out that the sum of a Fredholm operator with a strictly singular operator is always a Fredholm operator (cf. [1, Theorem 4.63]). In particular, strictly singular operators are Riesz.

It was proved in [8] that on any Banach lattice E, if $0 \le S \le T : E \to E$, with T strictly singular, then S^4 is strictly singular. Therefore, in this case, the dominated operator S is also Riesz. A bit more can actually be said:

Corollary 3.1. *Let $0 \le S \le T : E \to E$, with T a Riesz operator. If for some $n \in \mathbb{N}$ the operator T^n is strictly singular, then S is Riesz.*

Proof. Let $n \in \mathbb{N}$ such that the operator T^n is strictly singular. By [8], it follows that S^{4n} is strictly singular, and in particular S is a Riesz operator. □

This motivates the following weaker version of Problem 1.1.

Problem 3.1. *Let $0 \le S \le T : E \to E$, with T a Riesz operator. If T (or T^*) commutes with a non-zero strictly singular operator, is then S Riesz?*

The optimal power in the domination theorem for strictly singular operators given in [8] is not known. However, the technique can be adapted to show that under the same hypotheses S^3 is inessential [22]. Recall that an operator $T \in \mathcal{L}(X)$ is inessential if $I + UT$ is a Fredholm operator for every $U \in \mathcal{L}(X)$. Hence, strictly singular operators are inessential, and these are in turn Riesz. Concerning the domination of Riesz operators, that the dominated operator has an inessential finite power is enough for our purposes.

Since the strictly singular operators $\mathcal{S}(E)$ form a closed ideal of $\mathcal{L}(E)$, we can consider the quotient algebra

$$\mathcal{C}_{\mathcal{S}}(X) = \mathcal{L}(X)/\mathcal{S}(X).$$

Let us denote the natural quotient map by $\pi_S : \mathcal{L}(X) \to \mathcal{C}_{\mathcal{S}}(X)$ which is in fact an algebra homomorphism.

As in the case of the essential spectrum, given an operator $T \in \mathcal{L}(X)$, one can also consider the spectrum of $\pi_{\mathcal{S}}(T)$ in the algebra $\mathcal{C}_{\mathcal{S}}(X)$:

$$\sigma_{\mathcal{S}}(T) = \{\lambda \in \mathbb{C} : \lambda I - \pi(T) \text{ is not invertible in} \, \mathcal{C}_{\mathcal{S}}(X)\}.$$

Note that in general, $\mathcal{C}(X) \neq \mathcal{C}_{\mathcal{S}}(X)$. In fact, these two algebras coincide precisely when the ideals of compact and strictly singular operators coincide on X (see for instance [11] for examples of spaces with this property). However, the spectrum of an operator in $\mathcal{C}_{\mathcal{S}}(X)$ always coincides with the essential spectrum:

Proposition 1. *Given $T \in \mathcal{L}(X,Y)$, the following are equivalent:*

1. *T is Fredholm.*
2. *There exist compact operators $K_1 \in \mathcal{K}(X)$, $K_2 \in \mathcal{K}(Y)$, and $R_1, R_2 \in \mathcal{L}(Y,X)$ such that $R_1T = I_X - K_1$ and $TR_2 = I_Y - K_2$.*
3. *$\pi_S(T)$ is invertible in $\mathcal{C}_{\mathcal{S}}(X)$.*
4. *There exist strictly singular operators $S_1 \in \mathcal{S}(X)$, $S_2 \in \mathcal{S}(Y)$, and $R_1, R_2 \in \mathcal{L}(Y,X)$ such that $R_1T = I_X - S_1$ and $TR_2 = I_Y - S_2$.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) is due to F. Atkinson (cf. [1, Theorem 4.46]). It is clear that (2) $\Rightarrow$ (4) $\Leftrightarrow$ (3). Let us see that (4) $\Rightarrow$ (1): Let $S_1 \in \mathcal{S}(X)$, $S_2 \in \mathcal{S}(Y)$, and $R_1, R_2 \in \mathcal{L}(Y,X)$ such that $R_1T = I_X - S_1$ and $TR_2 = I_Y - S_2$. Since $S_1 \in \mathcal{S}(X)$, by [1, Theorem 4.63], it follows that $\ker(R_1T)$ is finite dimensional. Thus, so is $\ker(T) \subset \ker(R_1T)$.

Similarly, since $S_2 \in \mathcal{S}(Y)$, by [1, Theorem 4.63], we have that TR_2 is Fredholm. Hence, there is a finite dimensional subspace Z of Y such that $TR_2(Y) \oplus Z = Y$. Since $TR_2(Y) \subset T(X)$, it follows that $T(X) + Z = Y$. Thus, there exists a (finite dimensional) subspace $Z_1 \subset Z$ such that $T(X) \oplus Z_1 = Y$. Therefore, T is Fredholm. □

In particular, we get the following immediate

Corollary 3.2. *For every operator $T : X \to X$ we have $\sigma_{ess}(T) = \sigma_{\mathcal{S}}(T)$. In particular, T is Riesz if and only if $\sigma_{\mathcal{S}}(T) = \{0\}$.*

Recall that an operator $T : E \to X$ between a Banach lattice E and a Banach space X is disjointly strictly singular (DSS) if it is not invertible when restricted to the span of any sequence of pairwise disjoint vectors. This class which contains the strictly singular operators has proved useful for understanding the properties of strictly singular operators on Banach lattices (see [8]).

Theorem 3.1. *Let $0 \leq T : L_p \to L_p$ be a DSS operator (for some $1 < p < \infty$). It follows that $T : L_p \to L_p$ is Riesz if and only if $T : L_r \to L_r$ is Riesz for every $1 < r < \infty$.*

Proof. Let us assume $T : L_p \to L_p$ is a Riesz operator. By [23, Theorem 5], for each scalar $\lambda \neq 0$ and every $1 < r < \infty$ we have $\ker(\lambda I_{L_r} - T) = \ker(\lambda I_{L_p} - T)$. Hence, $\ker(\lambda I_{L_r} - T)$ is finite dimensional in L_r.

Note that by [23, Lemma 1] and [17, Proposition 3.6.11], it follows that $T^*: L_{p'} \to L_{p'}$ is DSS (with $\frac{1}{p}+\frac{1}{p'}=1$.) Now, for any λ we have that

$$R(\lambda I_{L_r}-T)^{\perp}=\{f\in L_{r'}:\int fg=0\forall g\in R(\lambda I_{L_r}-T)\}=\ker(\lambda I_{L_{r'}}-T^*)=\ker(\lambda I_{L_{p'}}-T^*).$$

Thus, $R(\lambda I_{L_r}-T)$ has finite codimension for every $1<r<\infty$. Therefore, $T: L_r \to L_r$ is Riesz for every $1<r<\infty$. □

4. Order spectrum and order essential spectrum

A useful technique to get positive results concerning the domination problem for Riesz operators involves the order spectrum. This concept was introduced by H. Schaefer in the 70's (cf. [17, §4.5]). Given an operator $T\in\mathcal{L}^r(E)$ on a Dedekind complete Banach lattice E, its order spectrum is defined as

$$\sigma_o(T)=\{\lambda\in\mathbb{C}:\lambda I-T \text{ is not invertible in } \mathcal{L}^r(E)\}.$$

The order spectral radius is $r_o(T)=\max\{|\lambda|:\lambda\in\sigma_o(T)\}$. Note that $\sigma(T)\subseteq\sigma_o(T)$ and $r(T)\leq r_o(T)$, but since $(\lambda I-T)^{-1}$ may belong to $\mathcal{L}(E)\backslash\mathcal{L}^r(E)$, in general we have $\sigma(T)\subsetneq\sigma_o(T)$. However, we have the following well-known

Lemma 1. *If $T\in\mathcal{L}(E)$ is positive, then $r(T)=r_o(T)$.*

Proof. Observe that for $\lambda\in\mathbb{C}$ with $|\lambda|>r(T)$, we have $(\lambda I-T)^{-1}=\sum_{n=0}^{\infty}\lambda^{-n-1}T^n$, and for every $x\in E$

$$|(\lambda I-T)^{-1}x|=\Big|\sum_{n=0}^{\infty}\lambda^{-n-1}T^n x\Big|\leq\sum_{n=0}^{\infty}|\lambda|^{-n-1}T^n|x|=(|\lambda|I-T)^{-1}|x|.$$

Thus, we have $\|(\lambda I-T)^{-1}\|_r\leq\|(|\lambda|I-T)^{-1}\|$ which is finite as far as $|\lambda|>r(T)$. Hence, there is no $\lambda\in\sigma_o(T)$ such that $|\lambda|>r(T)$. □

There is an analogous construction to that of the essential spectrum, starting with the order spectrum, that yields the order essential spectrum. This was introduced by W. Arendt and A. R. Sourour in [4] and is described next. Given a Dedekind complete Banach lattice, let $\mathcal{K}^r(E)$ denote the closure in the r-norm of the ideal of finite rank operators. This is a closed ideal of the Banach algebra $\mathcal{L}^r(E)$ which allows us to consider the Calkin algebra

$$\mathcal{C}^r(E)=\mathcal{L}^r(E)/\mathcal{K}^r(E),$$

together with the natural quotient $\pi^r:\mathcal{L}^r(E)\to\mathcal{C}^r(E)$. The order essential spectrum of a regular operator T is now defined as

$$\sigma_{oe}(T)=\{\lambda\in\mathbb{C}:\lambda I-\pi^r(T)\,\text{is not invertible in}\,\mathcal{C}^r(E)\}.$$

Similarly, we define the order essential radius $r_{oe}(T)=\max\{|\lambda|:\lambda\in\sigma_{oe}(T)\}$. For the order essential radius, the following domination result holds (see [16]):

Theorem 4.1. *Let E be a Dedekind complete Banach lattice. If $0\leq S\leq T\in\mathcal{L}(E)$, then $r_{oe}(S)\leq r_{oe}(T)$.*

The following result, also given in [16], provides a sufficient condition for the monotonicity of the essential radius.

Theorem 4.2. *Let E be a Dedekind complete Banach lattice, and $0 \leq S \leq T \in \mathcal{L}(E)$. If $\sigma(T) = \sigma_o(T)$, then $r_{ess}(S) \leq r_{ess}(T)$.*

Proof. Since $\sigma(T) = \sigma_o(T)$, by [4] the unbounded connected component of $\mathbb{C}\backslash\sigma_{ess}(T)$ coincides with the unbounded connected component of $\mathbb{C}\backslash\sigma_{oe}(T)$. In particular, we get $r_{ess}(T) = r_{oe}(T)$.

On the other hand, we always have $\sigma_{ess}(S) \subset \sigma_{oe}(S)$, and $r_{ess}(S) \leq r_{oe}(S)$. These facts, together with Theorem 4.1, yield

$$r_{ess}(S) \leq r_{oe}(S) \leq r_{oe}(T) = r_{ess}(T).$$

□

Recall that an operator $T : E \to E$ is 1-concave if there is a constant $C > 0$ such that for any $(x_i)_{i=1}^n$ in E

$$\sum_{i=1}^n \|Tx_i\| \leq C \Big\| \sum_{i=1}^n |x_i| \Big\|.$$

Also, T is ∞-convex if there is a constant $C > 0$ such that for any $(x_i)_{i=1}^n$ in E

$$\Big\| \max_{1\leq i\leq n} |Tx_i| \Big\| \leq C \max_{1\leq i\leq n} \|x_i\|.$$

In some parts of the literature, 1-concave operators are also refered as *cone absolutely summing* while ∞-convex operators are called majorizing.

Corollary 4.1. *Let $0 \leq S \leq T \in \mathcal{L}(E)$. We have that $r_{ess}(S) \leq r_{ess}(T)$ in any of the following situations:*

1. *E is an AL-space.*
2. *E is an AM-space.*
3. *T is ∞-convex and E has property (P).*
4. *T is 1-concave and E has property (P).*
5. *For every $x \geq 0$ in E, there is $w \geq x$ with $T(E_w) \subset E_w$ such that $T|_{E_w}$ is weakly compact.*

Proof. In any of these situations we have $\sigma(T) = \sigma_o(T)$ and Theorem 4.2 (see [16, Corollary 2.10] and [17, Theorem 4.5.3]). □

Concerning Problem 1.1, parts (iii) and (iv) of the corollary above can be slightly improved. For the sake of generality let us introduce the following:

Definition 1. *A Banach lattice E has the Riesz domination property (RDP, in short) whenever $0 \leq S \leq T : E \to E$ with T a Riesz operator, then S is also a Riesz operator.*

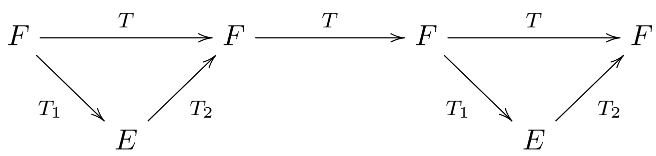
With this terminology Corollary 4.1 yields that AL-spaces and AM-spaces have the RDP. We will consider operators with the following (strong) factorization property:

Definition 2. *A positive operator $T : F \to F$ is Riesz factorable, if there exist a Banach lattice E with the RDP, and positive operators $T_1 : F \to E$, $T_2 : E \to F$ so that $T = T_2T_1$, and for each $0 \leq S \leq T$, there exist $0 \leq S_1 \leq T_1$ and $0 \leq S_2 \leq T_2$ with $S = S_2S_1$.*

For instance, the factorization results in [21] yield in particular that 1-concave and ∞-convex operators are Riesz factorable.

Proposition 2. *Suppose T is a Riesz factorable operator and $0 \leq S \leq T$, then S is Riesz.*

Proof. Let E with the RDP, and T_1, T_2 be as in the definition of a Riesz factorable operator T. Then we have the diagram



Note that $(T_1TT_2)^n = T_1T^nT_2$ for every $n \in \mathbb{N}$. Since T is a Riesz operator, we get

$$\|\pi(T_1TT_2)^n\|^{\frac{1}{n}} = \|\pi(T_1T^nT_2)\|^{\frac{1}{n}} \leq \|T_1\|^{\frac{1}{n}}\|\pi(T^n)\|^{\frac{1}{n}}\|T_2\|^{\frac{1}{n}} \underset{n\to\infty}{\longrightarrow} 0.$$

Hence, $T_1TT_2 : E \to E$ is also Riesz. Now, since $0 \leq S_1SS_2 \leq T_1TT_2$ and E has the RDP, it follows that S_1SS_2 is a Riesz operator. Thus $S^3 = S_2(S_1SS_2)S_1$ is Riesz, and so is S. □

5. An alternative approach: West decompositions

Let us start with the following simple observation. Suppose that an operator $T \in \mathcal{L}(X)$ can be written as $T = Q + K$, where Q is quasi-nilpotent in $\mathcal{L}(X)$ and $K \in \mathcal{K}(X)$. In this case, we have

$$r_{ess}(T) = \lim_{n\to\infty} \|\pi(T)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|\pi(Q)^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \|Q^n\|^{\frac{1}{n}} = 0.$$

Hence, T is a Riesz operator (see [15]). The interest in this decomposition arises from a result by T. West [25] which shows that the converse is also true for operators on Hilbert spaces. This result was later extended by K. Davidson and D. A. Herrero in [6] for operators in ℓ_p $(1 \leq p < \infty)$ and c_0.

By means of this decomposition together with some Banach lattice theory, we can provide an alternative proof for c_0. Note that this is a particular case of [20, Theorem 3.3].

Proposition 3. *Let $0 \leq R \leq T : c_0 \to c_0$. If T is a Riesz operator, then so is R.*

Proof. Let T be a Riesz operator. By [6], there exist operators K and Q in $\mathcal{L}(c_0)$, with K compact and Q quasinilpotent, such that $T = K + Q$.

Since $0 \leq S \leq T$ and $\mathcal{L}(c_0) = \mathcal{L}^r(c_0)$ is a Banach lattice, by the decomposition property [2, Thm 1.13] there exist K_1 and Q_1 in $\mathcal{L}(c_0)$, such that $S = K_1 + Q_1$, and $0 \leq K_1 \leq |K|$ and $0 \leq Q_1 \leq |Q|$.

By [14], it follows that $|K|$ is compact because so is K. Hence, by [7] $0 \leq K_1 \leq |K|$ implies that K_1 is also compact (since c_0 and c_0^* are order continuous Banach lattices). Moreover, since $0 \leq Q_1 \leq |Q|$ it is clear that Q_1 is quasinilpotent.

Therefore, we have

$$r_{ess}(S) = \lim \sqrt[n]{\|\pi(S)^n\|} \leq \lim \sqrt[n]{\|\pi(Q_1)^n\|} \leq \lim \sqrt[n]{\|Q_1^n\|} = 0.$$

□

Definition 3. *A positive operator T in a Banach lattice has a positive West decomposition if there exist positive operators K, Q such that K is compact, Q is quasinilpotent, and $T = K + Q$. If moreover, $QK = KQ$ then we say T has a positive West strong decomposition.*

Theorem 5.1. *Let E be a Dedekind complete Banach lattice. Given $0 \leq S \leq T : E \to E$, if T has a positive West strong decomposition, then S^3 has a positive West decomposition. In particular, S is a Riesz operator.*

Proof. Let K, Q be positive operators with $KQ = QK$ and $T = K + Q$. Since $0 \leq S \leq T$, and E is Dedekind complete, there exist K_1, Q_1 satisfying $S = K_1 + Q_1$, such that $0 \leq K_1 \leq K$ and $0 \leq Q_1 \leq Q$. Therefore we have $S^3 = K_1^3 + Q_2$, where

$$Q_2 = K_1^2 Q_1 + K_1 Q_1 K_1 + Q_1 K_1^2 + K_1 Q_1^2 + Q_1 K_1 Q_1 + Q_1^2 K_1 + Q_1^3.$$

Since $0 \leq K_1 \leq K$, by [3] we have that K_1^3 is compact. Now, using $0 \leq K_1 \leq K$, $0 \leq Q_1 \leq Q$, and $KQ = QK$, we get

$$r(Q_2) \leq r(3K^2Q + 3Q^2K + Q^3) \leq r(3K^2Q) + r(3Q^2K) + r(Q^3)$$

(cf. [18, Chapter I, Theorem 11]). Due to the commutativity of K and Q, each of the operators on the right hand side of the inequality is quasinilpotent, hence, so is Q_2. Therefore S^3 has a positive West decomposition, and S is Riesz. □

It is natural therefore to wonder whether positive operators with West decomposition must have a positive West decomposition. To the author's knowledge there are no known results in this direction.

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