ON THE DIRICHLET PROBLEM ON LORENTZ AND ORLICZ SPACES WITH APPLICATIONS TO SCHWARZ-CHRISTOFFEL DOMAINS

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ABSTRACT. It is known (see [14]) that, for every Lipschitz domain Ω on the plane

$$\Omega = \{x + iy : y > \nu(x)\},$$

with ν a real valued Lipschitz function, there exists $1 \leq p_0 < 2$ so that the Dirichlet problem has a solution for every function $f \in L^p(ds)$ and every $p \in (p_0, \infty)$. Moreover, if $p_0 > 1$, the result is false for every $p \leq p_0$. The purpose of this paper is to study in more detail what happens at the endpoint p_0; that is, we want to find spaces $X \subset L^{p_0}$ so that the Dirichlet problem is solvable for every $f \in X$. These spaces X will be either the Lorentz space $L^{p_0,1}(ds)$ or some type of logarithmic Orlicz space. Our results will be applied to the special case of Schwarz-Christoffel Lipschitz domains, among others, for which we explicitly compute the value of p_0.

1. INTRODUCTION AND MOTIVATION

Let Λ be a curve in the complex plane, given parametrically by $z(t) = t + i\nu(t)$, where ν is a real valued Lipschitz function with constant M and let us consider the Lipschitz domain

$$\Omega = \{x + iy : y > \nu(x)\}.$$

We shall examine some new aspect of the classical Dirichlet boundary value problem on Ω ([14, 10]),

$$\Delta u = 0, \qquad u_{|\Lambda} = f, \tag{1.1}$$

where

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2},$$

and $f \in C(\Lambda)$ is the data function. See also [21] for a related problem for other elliptic operators more general than the Laplacian Δ.

For every $1 \leq p < \infty$, we say that problem (1.1) is L^p-solvable if, for every $f \in C(\Lambda)$, the unique solution u satisfies that

$$\|M_\alpha u\|_{L^{p,\infty}(ds)} \leq C_\alpha \|f\|_{L^p(ds)}, \qquad 0 < \alpha < \arctan \frac{1}{M},$$

2010 *Mathematics Subject Classification.* 42B99, 46E30.
Key words and phrases. Dirichlet problem, Muckenhoupt weights, Yano's extrapolation, Lorentz spaces, Orlicz spaces.
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where, for every $z = a + ib \in \Lambda$,

$$M_\alpha u(z) = \sup_{x+iy\in\Gamma_\alpha(z)} |u(x+iy)|,$$

with $\Gamma_\alpha(z)$ the cone with axis in the vertical direction, vertex z and aperture α; that is,

$$\Gamma_\alpha(z) = \big\{x + iy \in \mathbb{C} : |x - a| \le (\tan\alpha)|y - b|\big\}.$$

It is known ([14]) that if $0 < \alpha < \arctan \frac{1}{M}$, it holds that $\Gamma_\alpha(z) \subset \Omega$.

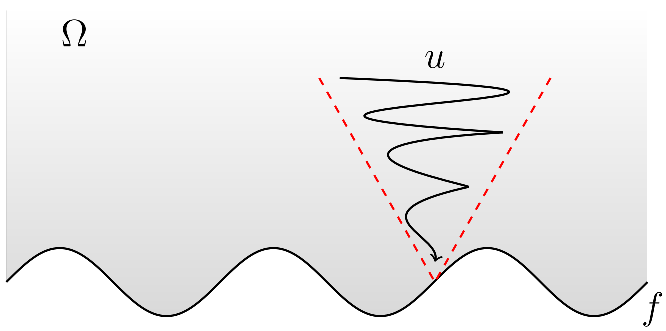

FIGURE 1. Non-tangencial convergence

Also, using standard techniques, it can be proved that, for every $f : \Lambda \to \mathbb{R}$ such that $f \in L^p(ds)$ with ds the arc length on Λ, there exists a harmonic function $u \in h(\Omega)$ so that u converges non-tangentially to f at almost every point $z \in \Lambda$,

$$\lim_{x+it\lhd z} u(x+iy) = f(z), \qquad \text{a.e. } z \in \Lambda,$$

where we use the notation $x + it \lhd z$ to indicate

$$\lim_{x+it\lhd z} u(x+iy) := \lim_{x+iy\to z, x+iy\in\Gamma_\alpha(z)} u(x+iy).$$

This notation will be use both in the setting of the upper half space $\mathbb{R}^2_+$ or in Ω.

Our starting motivation result is the following one due to C. Kenig in 1980 (see also [10]).

Theorem 1.1. [14] *Let ds be the arc length over Λ. Then, for every Lipschitz domain Ω, there exists $1 \le p_0 < 2$ so that, for every $p \in (p_0, \infty)$, the Dirichlet problem is L^p-solvable. Moreover, if $p_0 > 1$, the result is false for every $p \le p_0$.*

The idea of the proof was as follows. We refer to [14] for all the technical details. Let $\mathbb{R}^2_+ = \{x + iy : x > 0\}$ be the upper half space, $z_0 = ix_0$, $x_0 > \nu(0)$, and let $\Phi : \mathbb{R}^2_+ \to \Omega$ be the conformal mapping such that $\Phi(\infty) = \infty$ and $\Phi(i) = z_0$. Let Ψ be its inverse. Then, it holds that $\Phi(x)$ is absolutely continuous when restricted to any finite interval, and hence $\Phi'(x)$ exists almost everywhere, is locally integrable, and it holds that

$$\lim_{z\lhd x} \Phi'(z) = \Phi'(x), \qquad a.e.\ x \in \mathbb{R}.$$

The key point in [14] was to prove that

$$\sup_I \left(\frac{1}{|I|}\int_I |\Phi'(x)|\,dx\right)\left(\frac{1}{|I|}\int_I |\Phi'(x)|^{-1}dx\right) < \infty,$$

where the supremum is taking over all intervals of finite measure in $\mathbb{R}$. This condition says that $|\Phi'| \in A_2$, being A_p the classes of weights introduced by B. Muckenhoupt in [18] by the condition

$$\|w\|_{A_p} = \sup_I \frac{\int_I w(x)dx}{|I|}\left(\frac{\int_I w^{1-p'}(x)dx}{|I|}\right)^{p-1} < \infty, \qquad 1 < p < \infty,$$

and $u \in A_1$ if

$$Mu(x) \le Cu(x), \quad \text{a.e. } x \in \mathbb{R},$$

with $\|u\|_{A_1}$ being the least constant $C \ge 1$ that can be taken in such an inequality. Here, the operator M is the Hardy-Littlewood maximal operator, defined on locally integrable functions f by

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|}\int_I |f(y)|dy,$$

where the supremum is taken over all intervals $I \subseteq \mathbb{R}$ containing x. This class of weights are going to play an important role in this paper. In fact, it was also proved in [14] that if $w \in A_2$, there exists a Lipschitz domain Ω so that $w = |\Phi'|$, where Φ is the above conformal mapping, making a clear connection between Lipschitz domains and this class of weights.

By a simple change of variable, if $f \in L^2(ds)$, $(f \circ \Phi) \in L^2(|\Phi'|)$ and hence, if we solve the problem in $\mathbb{R}^2_+$ for every function in $L^2(|\Phi'|)$, we will solve the Dirichlet problem in our Lipstchitz domain for every function $f \in L^2(ds)$, by composing the function $u \in h(\mathbb{R}^2_+)$ with Ψ. Now, the solution of the Dirichlet problem in $\mathbb{R}^2_+$ is classical and is done with the help of the Poisson kernel; that is, if we define

$$u(x,t) = \int_{\mathbb{R}} P_t(x-y)f(y)dy = (P_t * f)(x)$$

with

$$P_t(y) = \frac{1}{\pi}\frac{t}{y^2+t^2}, \qquad y \in \mathbb{R},\ t > 0,$$

it is enough to consider the non-tangential maximal operator

$$P^*f(\xi) = \sup_{x+it \in \Gamma_\alpha(\xi)} |(P_t * f)(x)|,$$

where $\Gamma_\alpha(\xi)$ denotes a vertical cone in $\mathbb{R}^2_+$ with vertex at $\xi \in \mathbb{R}$, and prove that

$$P^* : L^2(|\Phi'|) \to L^2(|\Phi'|)$$

is bounded. On the other hand (see [23], Theorem 1, p. 197),

$$P^*f(\xi) \lesssim Mf(\xi),$$

and, if f is positive, $Mf \lesssim P^*f$. Therefore, the boundedness of P^* is equivalent to that of M, and it is known [18] that

$$M : L^p(w) \longrightarrow L^p(w), \qquad 1 < p < \infty,$$

is bounded if and only if $w \in A_p$, and

$$M : L^1(u) \longrightarrow L^{1,\infty}(u)$$

is bounded if and only if $u \in A_1$.

For later purposes, we include a list of the properties of A_p weights that are going to be important for our purposes. All of them can be found in Chapter 7 of [12].

i) Coifman-Rochberg characterization of A_1 weights: Every weight u in the class A_1 is of the form $u(x) = k(x)(Mf)^\delta$ where $k, k^{-1} \in L^\infty$, $f \in L^1_{loc}$ and $0 \leq \delta < 1$. Moreover, if the function $k(x) = 1$, then

$$\|(Mf)^\delta\|_{A_1} \leq \frac{C}{1-\delta}, \tag{1.2}$$

with C some universal constant independent of δ and f.
ii) P. Jones factorization theorem: A weight w is in the class A_p if and only if there exist two weights u_0 and u_1 in A_1 so that $w = u_0^{1-p} u_1$. Moreover,

$$\|u_0^{1-p} u_1\|_{A_p} \leq \|u_0\|_{A_1}^{p-1} \|u_1\|_{A_1}. \tag{1.3}$$

iii) Clearly, if $u \in A_p$, then, for every $q > p$, $u \in A_q$ and $\|u\|_{A_q} \leq \|u\|_{A_p}$.
iv) If $p > 1$ and $w \in A_p$, we have the so-called $p - \varepsilon$ property; that is, there exists $\varepsilon > 0$ so that $w \in A_{p-\varepsilon}$.

Using iv), the following result was proved in [14]: if p_0 is as in Theorem 1.1, then

$$p_0 = \inf\{q \geq 1 : |\Phi'| \in A_q\}. \tag{1.4}$$

We emphasize here that the main reason for the Dirichlet problem to be not solvable in L^{p_0} if $p_0 > 1$ is because, due to the $p - \varepsilon$-property, $|\Phi'|$ is never an A_{p_0} weight.

The first purpose of this paper is to study in more detail what happens near the endpoint p_0.

Definition 1.2. *We shall say that the Dirichlet problem is non-tangentially-solvable in X (and we shall write, in short, $\lhd$-solvable) if, for every $f \in X$, there exists a harmonic function $u \in h(\Omega)$ so that*

$$\lim_{x+iy \lhd z} u(x+iy) = f(z), \qquad a.e.\ z \in \Lambda. \tag{1.5}$$

Then, our goal is to find spaces $X \subset L^{p_0}(ds)$ so that the Dirichlet problem is $\lhd$-solvable in X. To attack this problem, the main idea will be "to measure" how far is the weight $|\Phi'|$ from the class A_{p_0}.

To this end, we need to recall the definition of two types of spaces: Lorentz spaces and Logarithmic type spaces. Given μ an arbitrary σ-finite measure, the Lorentz spaces $L^{p,1}(\mu)$ and $L^{p,\infty}(\mu)$ are defined as the space of measurable functions such that

$$||f||_{L^{p,1}(\mu)} = \int_0^\infty \lambda_f^\mu(y)^{\frac{1}{p}} dy = \frac{1}{p}\int_0^\infty f_\mu^*(t) t^{\frac{1}{p}-1} dt < \infty,$$

and, respectively,

$$||f||_{L^{p,\infty}(\mu)} = \sup_{y>0} y \lambda_f^\mu(y)^{1/p} = \sup_{t>0} t^{1/p} f_\mu^*(t) < \infty,$$

where f^*_μ is the decreasing rearrangement of f with respect to μ defined by

$$f^*_\mu(t) = \inf\{y > 0 : \lambda^\mu_f(y) \le t\}, \qquad \lambda^\mu_f(t) := \mu(\{x \in \Omega : |f(x)| > t\}).$$

Recall also that $L(\log L)^m(\mu) \subsetneq L^1(\mu)$ is the space of μ-measurable functions such that

$$\|f\|_{L(\log L)^m(\mu)} = \int_0^\infty f^*_\mu(t)\Big(1 + \log_+\Big(\frac{1}{t}\Big)\Big)^m dt < \infty,$$

where, as usual, $\log_+$ denotes the positive part of the logarithm. We will also need to consider other $\log$-type spaces. For simplicity, we will adopt the following notation:

$$\log_1(x) = 1 + \log_+(x) \quad \text{and} \quad \log_k(x) = \log_1 \log_{k-1}(x), \text{ for } k > 1.$$

In particular, $L(\log L)^m \log_3 L(\mu)$ is defined by the condition

$$\|f\|_{L(\log L)^m \log_3 L(\mu)} = \int_0^\infty f^*_\mu(t)\Big(\log_1\Big(\frac{1}{t}\Big)\Big)^m \log_3\Big(\frac{1}{t}\Big) dt < \infty.$$

The paper is organized as follows: in Section 2 we study the case in which (1.5) holds for every $f \in L^{p_0,1}(ds)$ and Section 3 will be devoted to the case on which X is a logarithmic Orlicz space. Along the paper, we shall present several examples of Lipschitz domains and give the precise behaviour near L^{p_0}. In particular, we pay special attention to the case of Schwarz-Christoffel domains, for which the value of p_0 is explicitly computed.

Let us mention that we write $x \lesssim y$ when there is a positive constant $C > 0$ such that $x \le Cy$. If both $x \lesssim y$ and $y \lesssim x$, then we write $x \approx y$. The constants involved do not depend on any parameter that is not fixed in its context. We shall also use the standard notation $u(E) = \int_E u(x)\, dx$ (if $u = 1$ we simply write $|E|$).

2. The Dirichlet problem on $L^{p_0,1}(ds)$

In this section, we want to study the case when the Dirichlet problem is $\lhd$-solvable in $L^{p_0,1}(ds)$, with $p_0 > 1$. To this end, we need to recall an important result due to R. Kerman and A. Torchinsky in [15], and independently by H. Chung, R. Hunt, D. Kurtz [9]:

Proposition 2.1.

$$M : L^{p,1}(w) \longrightarrow L^{p,\infty}(w)$$

is bounded if and only if $w \in A^{\mathcal{R}}_p$ defined by the condition

$$\|w\|_{A^{\mathcal{R}}_p} := \sup_{E \subset I} \frac{|E|}{|I|}\Big(\frac{w(I)}{w(E)}\Big)^{1/p} < \infty, \tag{2.1}$$

where the supremum is taking over all intervals I and every measurable set $E \subset I$. Moreover,

$$\|M\|_{L^{p,1}(w) \to L^{p,\infty}(w)} \lesssim \|w\|_{A^{\mathcal{R}}_p}. \tag{2.2}$$

It was proved in [9] that condition (2.1) is equivalent to

$$\|\chi_I\|_{L^{p,1}(w)} \|w^{-1}\chi_I\|_{L^{p',\infty}(w)} \lesssim |I|$$

for every interval I. This class of weights satisfies that, for every $\varepsilon > 0$,

$$A_p \subset A^{\mathcal{R}}_p \subset A_{p+\varepsilon},$$

and it is strictly bigger than A_p since it contains the weight $|x|^{(p-1)}$ which is not in A_p. Moreover, our example (3.1) shows that $A_p^{\mathcal{R}}$ is strictly smaller than $\cap_{\varepsilon>0} A_{p+\varepsilon}$. Hence, if p_0 is defined as in (1.4), it also holds that

$$p_0 = \inf\left\{q \geq 1 : |\Phi'| \in A_q^{\mathcal{R}}\right\}, \tag{2.3}$$

but there is an important difference between p_0 defined as in (1.4) or (2.3): If $p_0 > 1$, $|\Phi'|$ is never a weight in A_{p_0} but it could be that $|\Phi'| \in A_{p_0}^{\mathcal{R}}$. In this section, we shall consider the case where

$$|\Phi'| \in A_{p_0}^{\mathcal{R}} \setminus A_{p_0}, \qquad p_0 > 1.$$

Some of the properties of this class of weights that we are going to use in what follows are the following (see [4]):

i) For every $1 \leq p < \infty$,

$$\|w\|_{A_p^{\mathcal{R}}} \leq \|w\|_{A_p}^{1/p}. \tag{2.4}$$

ii) For every locally integrable function f (or even any finite measure μ) and every $u \in A_1$,

$$w = u(Mf)^{1-p} \in A_p^{\mathcal{R}}, \qquad v = u(M\mu)^{1-p} \in A_p^{\mathcal{R}}. \tag{2.5}$$

In particular, for every $u \in A_1$, $v(x) = u(x)|x|^{p-1} \in A_p^{\mathcal{R}}$, since $(M\delta)(x) = |x|^{-1}$.

iii) $|x|^{\beta} \in A_p^{\mathcal{R}}$ if and only if $-1 < \beta \leq p-1$.

iv) $A_1^{\mathcal{R}} = A_1$ and hence, if $|\Phi'| \in A_1^{\mathcal{R}}$, the Dirichlet problem is $\lhd$-solvable in $L^1(ds)$.

Our first main result can be stated as follows:

Theorem 2.2. *If $|\Phi'| \in A_{p_0}^{\mathcal{R}}$, the Dirichlet problem is non-tangentially-solvable in $L^{p_0,1}(ds)$.*

Proof. By Proposition 2.1, we have that

$$M : L^{p_0,1}(|\Phi'|) \longrightarrow L^{p_0,\infty}(|\Phi'|)$$

is bounded and hence, same estimate holds por P^*. Since, for every bounded function with compact support g, we know that

$$\lim_{x+iy \lhd \xi} (P_t * g)(x) = g(\xi), \qquad a.e.\ \xi \in \mathbb{R}, \tag{2.6}$$

and this collection of functions are dense in $L^{p_0,1}(|\Phi'|)$, we have by standard techniques that (2.6) holds for every $g \in L^{p_0,1}(|\Phi'|)$; that is, there exists a harmonic function $v \in h(\mathbb{R}^2_+)$ so that

$$\lim_{x+iy \lhd \xi} v(x+iy) = g(\xi), \qquad \text{a.e. } \xi \in \mathbb{R}.$$

Now, if $f : \Lambda \to \mathbb{R}$ and $f \in L^{p_0,1}(ds)$, then since

$$ds(\{x \in \Lambda : |f(x)| > y\}) = |\Phi'|(\{t \in \mathbb{R} : |f \circ \Phi(t)| > y\}), \tag{2.7}$$

we have that $f \circ \Phi \in L^{p_0,1}(|\Phi'|)$ and hence, there exists an harmonic function $v \in h(\mathbb{R}^2_+)$ so that

$$\lim_{x+iy \lhd\ \xi} v(x+iy) = (f \circ \Phi)(\xi), \qquad \text{a.e. } \xi \in \mathbb{R}.$$

Now, if we consider, for each $z \in \Omega$, $u(z) = (v \circ \Psi)(z)$, we have that $u \in h(\Omega)$ and it was proved in [14] that

$$\lim_{x+iy \lhd z} u(x+iy) = f(z), \qquad \text{a.e. } z \in \Lambda,$$

as we wanted to see. □

Let us analyze now when the hypothesis of Theorem 2.2 holds. By definition (I will always be an interval and E a measurable set),

$$|||\Phi'|||_{A^{\mathcal{R}}_{p_0}} = \sup_{E \subset I} \frac{|E|}{|I|} \Big(\frac{|\Phi(I)|}{|\Phi(E)|} \Big)^{1/p_0} < \infty,$$

and hence, if we define

$$\overline{\Phi}(t) := \sup \left\{ \frac{|\Phi(I)|}{|\Phi(E)|} : E \subseteq I \text{ and } |I| = t|E| \right\}, \tag{2.8}$$

we have that, $|\Phi'| \in A^{\mathcal{R}}_{p_0}$ if and only if

$$\overline{\Phi}(t) \lesssim t^{p_0}, \qquad \forall t > 1.$$

Now, it is known (see [17]) that $\overline{\Phi}$ is submultiplicative in the sense that for every $s, t > 1$,

$$\overline{\Phi}(st) \leq \overline{\Phi}(s)\overline{\Phi}(t),$$

and hence the following lemma will be useful.

Lemma 2.3. *(see* [1]*) If $\varphi : [1, \infty) \to [0, \infty)$ is a submultiplicative function and $0 < q < \infty$ then,*

(1) either $t^q \leq \varphi(t)$, for every $t > 1$, or
(2) there exists $\varepsilon > 0$ so that $\varphi(t) \lesssim t^{q-\varepsilon}$.

Corollary 2.4. *If $\overline{\Phi}$ is defined as in* (2.8) *and p_0 as in* (1.4)*, we have that*

$$p_0 = \lim_{t \to +\infty} \frac{\log \overline{\Phi}(t)}{\log t}. \tag{2.9}$$

Proof. First of all, due to the submultiplicative property of $\overline{\Phi}$, the above limit exists ([17]). Let $q := \lim_{t \to +\infty} \frac{\log \overline{\Phi}(t)}{\log t}$ and assume that $q < p_0$, then there exists $t_0 > 1$ so that

$$\log \overline{\Phi}(t_0) < p_0 \log t_0,$$

and hence $\overline{\Phi}(t_0) < t_0^{p_0}$ which implies by the previous lemma that, for some $\varepsilon > 0$, $\overline{\Phi}(t) \lesssim t^{p_0 - \varepsilon}$, for every $t > 1$. Hence $|\Phi'| \in A^{\mathcal{R}}_{p_0 - \varepsilon}$ and thus $|\Phi'| \in A_{p_0}$ which is a contradiction. On the other hand, by (2.3), there exists a sequence $p_n > p_0$ so that $\lim_n p_n = p_0$ and $\overline{\Phi}(t) \lesssim t^{p_n}$. Hence $\log \overline{\Phi}(t) \leq \log C + p_n \log t$ and thus $q \leq p_n$ for every n and taking the limit in n we obtain that $q \leq p_0$ and the result follows. □

Corollary 2.5. *Let p_0 be defined as in* (1.4) *(equivalently as in* (2.3) *or* (2.9)*). Then, $|\Phi'| \in A^{\mathcal{R}}_{p_0}$ if and only if*

$$\overline{\Phi}(t) \approx t^{p_0}, \qquad \forall t > 1.$$

Remark 2.6. *If $\varphi : [1,\infty) \to [0,\infty)$ is a submultiplicative function and*

$$q := \lim_{t\to+\infty} \frac{\log \varphi(t)}{\log t},$$

it is not true, in general, that $\varphi(t) \lesssim t^q$, as the example, $\varphi(t) = t^q(1+\log t)$ shows. Therefore, in our context, p_0 can be computed by (2.9) *but this does not imply, in general, that $|\Phi'| \in A_{p_0}^{\mathcal{R}}$.*

2.1. **The Dirichlet problem in Schwarz-Christoffel Lipschitz domains.**

Let $x_0 \in \mathbb{R}$ be fixed, and let us consider $\Phi : \mathbb{R}^2_+ \to \mathbb{C}$ with derivative

$$\Phi'(z) = (z - x_0)^{\alpha-1}, \qquad 0 < \alpha < 2.$$

Here we choose the branch of the argument so that

$$-\frac{\pi}{2} < \arg(z - x_0) \leq \frac{3\pi}{2},$$

introducing a branch cut along the axis $\{x_0 + iy : y \leq 0\}$. In this way, Φ is clearly analytic in $\mathbb{R}^2_+$.

(1) Let Ω be a cone with aperture $\alpha\pi > 0$ as in Figures 2 or 3. Then, Ω is a Lipschitz domain and, for some $A \in \mathbb{C}$, $\Phi(z) = Az^\alpha$ is the conformal map $\Phi : \mathbb{R}^2_+ \to \Omega$. In this case, $|\Phi'(x)| = A\alpha|x|^{\alpha-1}$. We observe two different behaviours depending on α:

a) If $\alpha \leq 1$, we have that $|\Phi'(x)| \in A_1$, and hence, the Dirichlet problem is solvable in $L^1(ds)$. This is the case of Figure 2.

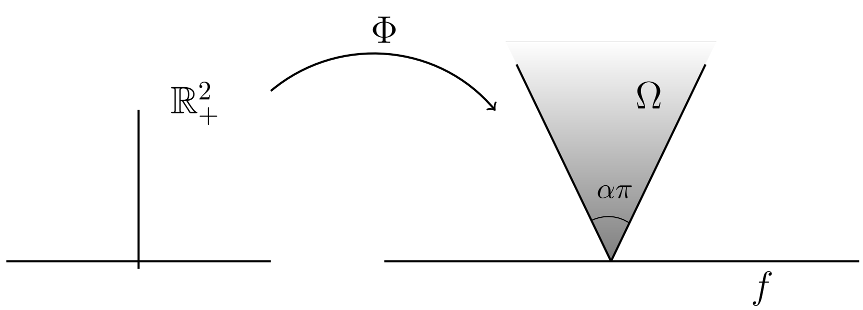

FIGURE 2. Cone with aperture $\alpha\pi$ with $0 < \alpha < 1$

b) On the contrary, if $1 < \alpha < 2$, $|\Phi'(x)| \in A_p^{\mathcal{R}}$ if and only if $\alpha \leq p$ and hence $p_0 = \alpha$, and $|\Phi'(x)| \in A_\alpha^{\mathcal{R}} \setminus A_\alpha$. Consequently, the Dirichlet problem is $\lhd$-solvable in $L^{\alpha,1}(|\Phi'|)$ but not in $L^\alpha(|\Phi'|)$. This is the case of Figure 3.

(2) Let us now consider the domain Ω as in Figure 4. In this case, the conformal mapping $\Phi : \mathbb{R}^2_+ \to \Omega$ is given by (see [13])

$$\Phi(z) = ie^{i\frac{\pi}{4}}\Big(\arccos z + \sqrt{1-z^2}\Big).$$

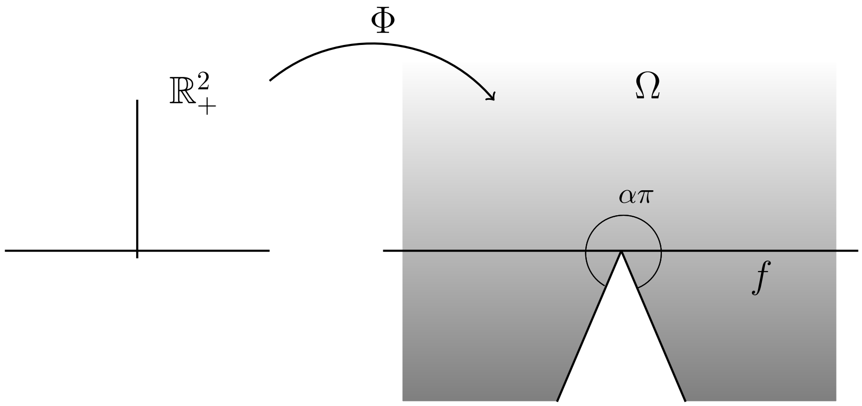

FIGURE 3. Cone with aperture $\alpha\pi$ with $1 < \alpha < 2$

Then,

$$\Phi'(x) = ie^{i\frac{\pi}{4}}\left(\frac{1}{\sqrt{1-x^2}} + \frac{-x}{\sqrt{1-x^2}}\right) = ie^{i\frac{\pi}{4}}(1-x)^{1/2}(1+x)^{-1/2},$$

and hence, by (2.5),

$$|\Phi'(x)| = |1-x|^{1/2}|1+x|^{-1/2} \in A^{\mathcal{R}}_{3/2} \setminus A_{3/2}.$$

Consequently $p_0 = \frac{3}{2}$ and the Dirichlet problem in this domain is $\lhd$-solvable in $L^{\frac{3}{2},1}(ds)$ but not in $L^{3/2}(ds)$.

All these examples are, in fact, particular cases of Schwarz-Christoffel domains, which we shall develop next in more detail.

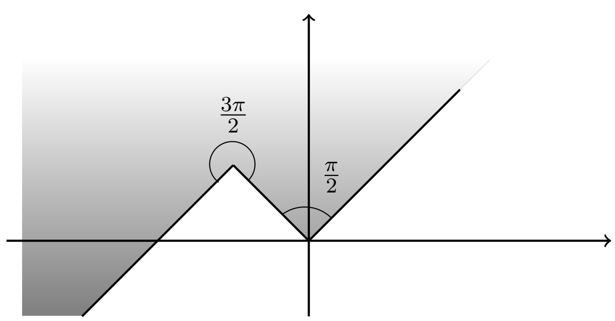

FIGURE 4. Two angles domain

Definition 2.7. *A function $\Phi : \mathbb{R}^2_+ \to \mathbb{C}$ is called a Schwarz-Christoffel transformation if, for some complex numbers A and B,*

$$\Phi(z) = A + B\int_{[z_0,z]}(\zeta - x_1)^{\alpha_1 - 1}\cdots(\zeta - x_n)^{\alpha_n - 1}\,d\zeta, \tag{2.10}$$

where z_0 is a suitably chosen point in $\mathbb{R}^2_+$ or its boundary, $[z_0, z]$ denotes the straight line segment from z_0 to z, $\{x_1 < \cdots < x_{n-1}\} \subset \mathbb{R}$ and $0 < \alpha_1, \ldots \alpha_{n-1} < 2$.

Definition 2.8. *We say that Ω is a Polygon if its boundary Λ is a polygonal with a finite number of vertices.*

Theorem 2.9. *([11]) Suppose that Ω is a Polygon with vertices $w_1, \ldots, w_n$ with right turns angles in the anticlockwise direction $\alpha_1\pi, \ldots \alpha_n\pi$, where $0 < \alpha_1, \ldots, \alpha_{n-1} < 2$, and its verifies that*

$$\sum_{i=1}^{n} \alpha_i = n - 2. \tag{2.11}$$

Then there exists a Schwarz-Christoffel transformation (2.10) that maps $\mathbb{R}^2_+$ one-to-one and conformally onto the interior of P, with $\Phi(x_i) = w_i, \forall i = 1, \ldots, n-1$, $\Phi(\infty) = w_n$. Moreover, if $w_n = \infty$, (as in Figure 5), then $\alpha_n \in (-2, 0)$.

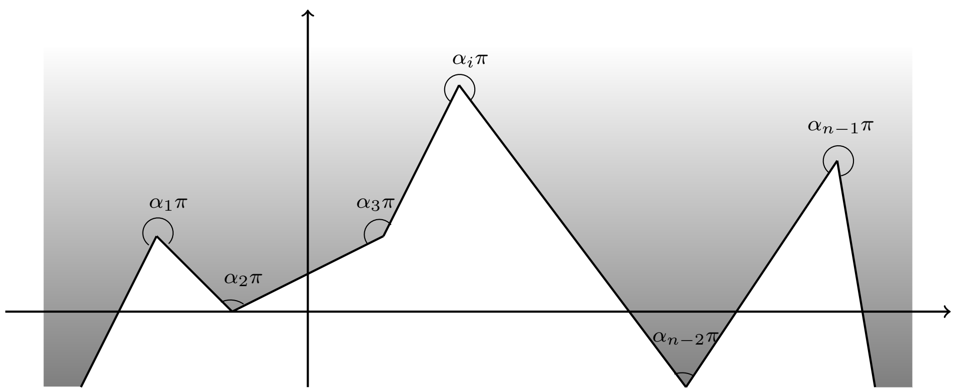

FIGURE 5. General case

Since

$$|\Phi'(x)| \approx |x - x_1|^{\alpha_1 - 1} \cdots |x - x_{n-1}|^{\alpha_{n-1}},$$

to analyze in which class of weights is $|\Phi'|$ we need the following lemma (see [19] for related results):

Lemma 2.10. *If*

$$w(x) = |x - x_1|^{\alpha_1 - 1}|x - x_2|^{\alpha_2 - 1} \cdots |x - x_{n-1}|^{\alpha_{n-1} - 1},$$

with $\alpha_i > 0$ for any i, then $w \in A_q^{\mathcal{R}}$, with

$$q = \max_{i=1,\ldots,n-1} \Big\{ \alpha_i, -n + \sum_{j=1}^{n-1} \alpha_j \Big\}. \tag{2.12}$$

Moreover, $w \notin A_p^{\mathcal{R}}$ for any $p < q$.

Proof. We shall assume that $x_1 < x_2 < \cdots < x_{n-1}$ and set $N \in \mathbb{N}$ fixed such that $-N < x_1 < x_{n-1} < N$. Let us define $d_i = x_{i+1} - x_i$ and $d = \min_i d_i$. Recall that $|x|^\alpha \in A_p^{\mathcal{R}}$ if and only if $p \geq \alpha + 1$.

In order to check that $w \in A_q^{\mathcal{R}}$ we have to prove that, for every interval I,

$$\|\chi_I\|_{L^{q,1}(w)} \|w^{-1}\chi_I\|_{L^{q',\infty}(w)} \lesssim |I|. \tag{2.13}$$

Thus, let us consider an arbitrary interval $I \subset \mathbb{R}$ and let us analyze the following cases:

(1) If $I \cap [-2N, 2N] = \emptyset$, then, for every $x \in I$ and every i, $|x - x_i| \approx |x|$ and hence, $w(x) \approx |x|^{\sum_{i=1}^{n-1}(\alpha_i - 1)}$. This implies that if $w \in A_q^{\mathcal{R}}$ necessarily $q \geq -n + \displaystyle\sum_{i=1}^{n-1} \alpha_i$. Moreover, if q satisfies this inequality, (2.13) holds for such an interval I.

(2) If $I \cap [-2N, 2N] \neq \emptyset$ and $|I| < \frac{d}{2}$, then there exists i so that, for $x \in I$, $w(x) \approx |x - x_i|^{\alpha_i - 1}$. Hence, if $w \in A_q^{\mathcal{R}}$ necessarily $q \geq \alpha_i$. Consequently, if $w \in A_q^{\mathcal{R}}$ necessarily $q \geq \displaystyle\max_{i=1,\dots,n-1} \alpha_i$, and for these intervals and these values of q, (2.13) holds.

(3) $I \cap [-2N, 2N] \neq \emptyset$ and $|I| \geq \frac{d}{2}$, we can assume, without lost of generality, I is a symmetric interval containing $[-2N, 2N]$; that is $[-2N, 2N] \subset I = (-b, b)$. Then,

$$\begin{aligned}
w(I) &= \int_{-b}^{b} w(x)\,dx = \int_{-b}^{b} |x - x_1|^{\alpha_1 - 1}|x - x_2|^{\alpha_2 - 1} \cdots |x - x_{n-1}|^{\alpha_{n-1} - 1}\,dx \\
&\approx \int_{-b}^{-2N} |x|^{\sum_{i=1}^{n-1}(\alpha_i - 1)}\,dx + \int_{-2N}^{x_1 + d_1/2} |x - x_1|^{\alpha_1 - 1}\,dx + \int_{x_2 - d_1/2}^{x_2 + d_2/2} |x - x_2|^{\alpha_2 - 1}\,dx \\
&+ \cdots + \int_{x_{n-2} - d_{n-3}/2}^{x_{n-2} + d_{n-3}/2} |x - x_{n-2}|^{\alpha_2 - 1}\,dx + \int_{x_{n-1} - d_{n-2}/2}^{2N} |x - x_{n-1}|^{\alpha_{n-1} - 1}\,dx \\
&+ \int_{2N}^{b} |x|^{\sum_{i=1}^{n-1}(\alpha_i - 1)}\,dx \approx 1 + \int_{2N}^{b} |x|^{\sum_{i=1}^{n-1}(\alpha_i - 1)}\,dx.
\end{aligned}$$

From here, it follows that if $b - 2N \leq N$, then $w(I)$ is controlled by a constant (depending on N, which is fixed and depending only on $\{x_i\}_i$) and if $b - 2N > N$, then $w(I)$ is controlled by $v(I \cap [2N, \infty))$ with $v(x) = |x|^{\sum_{i=1}^{n-1}(\alpha_i - 1)}$.

On the other hand, with similar computations, it is easy to see that

$$\|w^{-1}\chi_I\|_{L^{q',\infty}(w)} \approx 1 + \|v^{-1}\chi_{I \cap [2N,\infty)}\|_{L^{q',\infty}(v)},$$

and hence, if $b - 2N \leq N$,

$$w(I)^{1/q} \|w^{-1}\chi_I\|_{L^{q',\infty}(w)} \lesssim 1 \leq \frac{2}{d}|I| \lesssim |I|,$$

and if $b - 2N > N$ and $q \geq -n + \displaystyle\sum_{i=1}^{n-1} \alpha_i$,

$$w(I)^{1/q} \|w^{-1}\chi_I\|_{L^{q',\infty}(w)} \lesssim v(I \cap [2N, \infty))^{1/q} \|v^{-1}\chi_{I \cap [2N,\infty)}\|_{L^{q',\infty}(v)} \lesssim |I|;$$

that is, (2.13) holds for this type of intervals. Putting all the cases together, we conclude the result. $\square$

Corollary 2.11. *If Ω is a Schwarz-Christoffel domain as in Figure 4, the Dirichlet problem is $\lhd$-solvable in $L^{q,1}(ds)$ with $q = \max(\alpha_i)$.*

Proof. By Theorem 2.9 and (2.11),

$$-n + \sum_{i=1}^{n-1} \alpha_i = -2 - \alpha_n < 0,$$

and hence the result follows by (2.12). □

Let us see other examples of domains where Λ is a polygonal with infinite vertices. For example, let us consider an infinite rotated stairway with interior angles alternately equal to $\pi/2$ and $3\pi/2$ as in Figure 6. The formula in this case is (see [20])

$$\Phi(z) = A + B \int_0^z \sqrt{\tan \xi}\, d\xi.$$

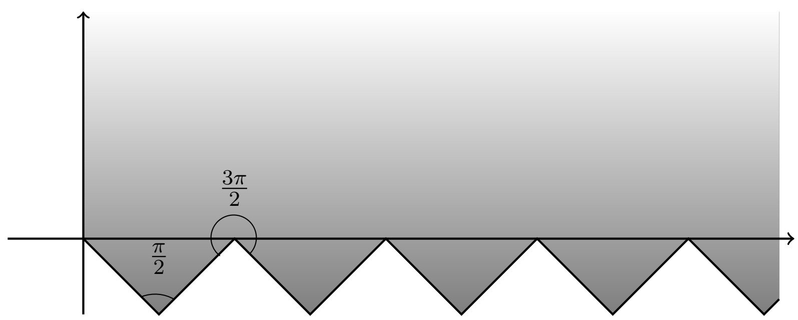

FIGURE 6. Infinite domain with interior angles $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ alternately

Thus, $|\Phi'(x)| = |B\sqrt{\tan x}| \approx |\sin x|^{1/2} |\cos x|^{-1/2}$. Now, a π-periodic function $w \in A_p$ if and only if

$$\sup_I \left(\frac{1}{|I|} \int_I |\Phi'(x)|\, dx \right) \left(\frac{1}{|I|} \int_I |\Phi'(x)|^{-1} dx \right) < \infty,$$

where the supremum is taking over all intervals in $[-\pi, \pi]$ and similarly for $w \in A_p^{\mathcal{R}}$. Hence, if $p(f)$ is the 2π-periodic extension of f and since

$$|\sin x|^{1/2} = \left(\frac{|\sin x|}{|p(x)||p(x-\pi)|} \right)^{1/2} \Big| p(x)p(x-\pi) \Big|^{1/2} := h_1(x) \Big| p(x)p(x-\pi) \Big|^{1/2},$$

with $h_1, h_1^{-1} \in L^\infty$, and similary

$$|\cos x|^{-1/2} = h_2(x) \Big| p(x - \frac{\pi}{2}) p(x + \frac{\pi}{2}) \Big|^{-1/2}$$

with $h_2, h_2^{-1} \in L^\infty$, we have that

$$|h(x)| \approx \Big| x(x-\pi) \Big|^{1/2} \Big| (x - \frac{\pi}{2})(x + \frac{\pi}{2}) \Big|^{-1/2}.$$

Now, following the same computations as in Lemma 2.10 (see also [19]), we have that $p_0 = \max(\frac{3}{2}, \frac{1}{2})$ and hence $|\Phi'| \in A^{\mathcal{R}}_{3/2} \setminus A_{3/2}$. Thus, the Dirichlet problem is $\lhd$-solvable in $L^{3/2,1}(ds)$ but not in $L^{3/2}(ds)$.

More generally, if we consider the conformal map (see [16])

$$\Phi(z) = A + B\int_0^z (1-\cos\xi)^{\frac{\alpha_1-1}{2}}(1+\cos\xi)^{\frac{\alpha_n-1}{2}}\prod_{i=2}^{n-1}(\cos x_i - \cos\xi)^{\alpha_i-1}d\xi,$$

with $x_i \in \mathbb{R}$ and $\alpha_i \in (0,2)$, the image $\Phi(\mathbb{R}^2_+) = \Omega$ is a domain with a countable number of vertices $w_i \in \Omega$ with pre-images x_i and angles $\alpha_i\pi$ at w_i, of the type of Figure 7. In this case,

$$\begin{aligned} |\Phi'(x)| &\approx \left|(1-\cos x)^{\frac{\alpha_1-1}{2}}(1+\cos x)^{\frac{\alpha_n-1}{2}}\prod_{i=2}^{n-1}(\cos x_i - \cos x)^{\alpha_i-1}\right| \\ &\approx \left|x^{\alpha_1-1}(x-\pi)^{\alpha_n-1}\prod_{i=2}^{n-1}|x-x_i|^{\alpha_i-1}\right| \in A^{\mathcal{R}}_{p_0}\setminus A_{p_0} \end{aligned}$$

where $p_0 = \max_i \alpha_i$.

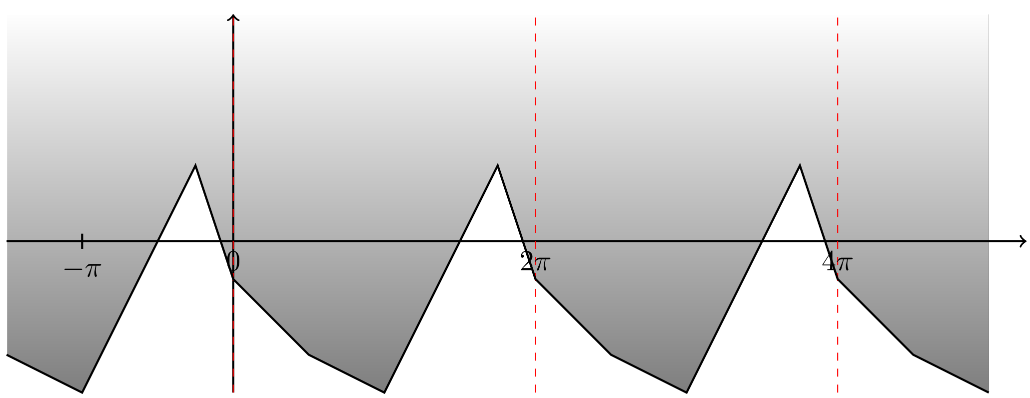

FIGURE 7. Infinite 2π periodic domain

3. LOGARITHMIC TYPE SPACES

As mentioned before, it is not true, in general, that if p_0 is defined as in (2.3), it holds that $|\Phi'| \in A^{\mathcal{R}}_{p_0}$. To see this let us consider the case of

$$|\Phi'(x)| = \frac{|x|^{r-1}}{1+\log_+\frac{1}{|x|}}, \qquad r > 1. \tag{3.1}$$

In this case, for every $t > 1$,

$$\overline{\Phi}(t) \geq \sup_{0<y<1} \frac{\int_0^y |\Phi'(x)|dx}{\int_0^{y/t} |\Phi'(x)|dx} \approx t^r \sup_{0<y<1} \frac{1+\log_+ \frac{t}{y}}{1+\log_+ \frac{1}{y}},$$

and hence, it does not hold that $\overline{\Phi}(t) \lesssim t^r$. Therefore, $|\Phi'| \notin A_r^{\mathcal{R}}$. On the other hand, let us take $r < p < r + \frac{1}{2}$, $0 < \delta < 1$ so that $r < 1 + (p-1)\delta = \frac{p+r}{2}$ and let $r < q < 1 + (p-1)\delta$,

$$f(x) = \frac{1}{|x|^{\frac{q-1}{(p-1)\delta}}} \quad \text{and} \quad g(x) = \frac{1}{|x|^{2(q-r)}(1+\log_+ \frac{1}{|x|})^2}.$$

Since these functions are locally integrable, even and equivalent to a decreasing function, we have that $Mf \approx f$ and $Mg \approx g$. Thus,

$$\frac{|x|^{r-1}}{1+\log_+ \frac{1}{|x|}} = |x|^{q-1} . \frac{|x|^{r-q}}{1+\log_+ \frac{1}{|x|}} \approx (Mf)^{\delta(1-p)}(Mg)^{1/2},$$

and hence, by properties i) and ii) listed in the introduction, $|\Phi'| \in A_p$, for every $p > r$, which implies that $p_0 = r$ in this case and $|\Phi'| \notin A_{p_0}^{\mathcal{R}}$ as we wanted to see.

Moreover, by (1.2) and (1.3),

$$\||\Phi'|\|_{A_p} \lesssim \left(\frac{1}{1-\delta}\right)^{p-1} \approx \left(\frac{1}{p-r}\right)^{p-1} \approx \left(\frac{1}{p-r}\right)^{p_0-1}.$$

This is the situation we want to deal with in this section; that is, we shall mainly study the case when $|\Phi'|$ is not in $A_{p_0}^{\mathcal{R}}$ but we have some information about the behaviour of $\||\Phi'|\|_{A_p}$ or $\||\Phi'|\|_{A_p^{\mathcal{R}}}$ when $p \to p_0^+$. We study several cases:

Case I: Let us assume that $p_0 = 1$ and that the following condition holds:

$$\||\Phi'|\|_{A_p^{\mathcal{R}}} \lesssim (p-1)^{-m}.$$

In this case, the following lemma will be fundamental. In 1996, N. Yu Antonov [2] proved that, for every function in $L \log L \log_3 L(\mathbb{T})$, the Fourier series converges at almost every point. Even though he did not write it explicitly, behind his ideas there was the following result (see [3, 6] for more details):

Lemma 3.1. *Let μ be a σ- finite measure space. If T is a sublinear operator such that*

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\mu)$$

is bounded with constant controlled by $(p-1)^{-m}$ for every $1 < p \leq 2$, then

$$T : L(\log L)^m \log_3 L(\mu) \longrightarrow R_m^{1,\infty}(\mu)$$

is bounded where

$$\|f\|_{R_m^{1,\infty}(\mu)} = \sup_{t>0} \frac{t f_\mu^*(t)}{(1+\log_+ t)^m}.$$

Theorem 3.2. *If $|\Phi'| \notin A_1$ but there exists $m > 0$ so that*

$$\||\Phi'|\|_{A_p^{\mathcal{R}}} \lesssim \frac{1}{(p-1)^m}, \qquad 1 < p \leq 2,$$

the Dirichlet problem is $\lhd$-solvable in $L(\log L)^m(\log_3 L)(ds)$.

Proof. By (2.2),

$$\|M\|_{L^{p,1}(|\Phi'|)\to L^{p,\infty}(|\Phi'|)} \lesssim \frac{1}{(p-1)^m},$$

and hence we can apply Lemma 3.1 to obtain that

$$M : L(\log L)^m(\log_3 L)(|\Phi'|) \longrightarrow R_m^{1,\infty}(|\Phi'|)$$

is bounded and so is P^*. Now, using the same technique than in Theorem 2.2 we observe first that bounded functions with compact support are dense in $L(\log L)^m(\log_3 L)(|\Phi'|)$ (see Theorem 2.3.12, [8]). On the other hand, $g \in R_m^{1,\infty}(|\Phi'|)$ if and only if

$$g^*_{|\Phi'|}(t) \lesssim \|g\|_{R_m^{1,\infty}(|\Phi'|)}\varphi(t), \qquad \forall t > 0,$$

where φ is a decreasing function such that $\varphi(t) \approx \frac{(1+\log_+ t)^m}{t}$. Equivalently,

$$|\Phi'|(\{x : |g(x)| > y\}) \lesssim \varphi^{-1}\left(\frac{y}{\|g\|_{R_m^{1,\infty}(|\Phi'|)}}\right),$$

and we observe that $\lim_{z\to\infty}\varphi^{-1}(z) = 0$. Now, for every $f \in L(\log L)^m(\log_3 L)(|\Phi'|)$ and every $\varepsilon > 0$, there exists a bounded function with compact support g so that $\|f - g\|_{L(\log L)^m(\log_3 L)(|\Phi'|)} < \varepsilon$, and since $g \in L^p(|\Phi'|)$ with $p > p_0$ and $|\Phi'| \in A_p$, it holds that

$$\lim_{x+it\lhd\xi}(P_t * g)(x) = g(\xi), \qquad \text{a.e. } \xi \in \mathbb{R}.$$

From here, it follows that, for every $\delta > 0$,

$$\begin{aligned}
&|\Phi'|(\{x : \overline{\lim}_{x+it\lhd\xi}|P_t * f(x) - f(x)| > \delta\}) \\
\lesssim\ & |\Phi'|(\{x : \overline{\lim}_{x+it\lhd\xi}|P_t * (f-g)(x) - (f-g)(\xi)| > \delta/2\}) \\
\lesssim\ & |\Phi'|(\{\xi : P^*(f-g)(\xi)| > \delta/4\}) \\
\lesssim\ & \varphi^{-1}\left(\frac{\delta}{4\|P^*(f-g)\|_{R_m^{1,\infty}(|\Phi'|)}}\right) \\
\lesssim\ & \varphi^{-1}\left(\frac{C\delta}{\|f-g\|_{L(\log L)^m(\log_3 L)(|\Phi'|}}\right) \lesssim \varphi^{-1}\left(\frac{C\delta}{\varepsilon}\right) \stackrel{\varepsilon\to 0}{\longrightarrow} 0,
\end{aligned}$$

and therefore,

$$\lim_{x+it\lhd\xi}(P_t * f)(x) = f(\xi), \qquad \text{a.e. } \xi \in \mathbb{R}, \qquad \forall f \in L(\log L)^m(\log_3 L)(|\Phi'|).$$

Finally, if $f : \Lambda \to \mathbb{R}$ satisfies that $f \in L(\log L)^m(\log_3 L)(ds)$, we have by (2.7) that $f \circ \Phi \in L(\log L)^m(\log_3 L)(|\Phi'|)$ and the result follows, as explained in Theorem 2.2. ☐

Using (2.4), we have that

$$|||\Phi'|||_{A_p^{\mathcal{R}}} \lesssim |||\Phi'|||_{A_p}^{1/p},$$

and hence, the following corollary is immediate.

Corollary 3.3. *If $|\Phi'| \notin A_1$ but there exists $m > 0$ so that*

$$|||\Phi'|||_{A_p} \lesssim (p-1)^{-m},$$

the Dirichlet problem is $\lhd$-solvable in $L(\log L)^m(\log_3 L)(ds)$.

Case II: Let us assume that $1 < p_0 < 2$ and

$$|||\Phi'|||_{A_p^{\mathcal{R}}} \lesssim (p-p_0)^{-m}, \qquad p_0 < p \leq 2. \tag{3.2}$$

To solve this case we need the following lemma.

Lemma 3.4. *([5, 7]) If T satisfies*

$$||Tf||_{L^{p,\infty}(\mu)} \lesssim \frac{1}{(p-p_0)^m}||f||_{L^{p,1}(\mu)}, \qquad \forall p \in (p_0, 2), \tag{3.3}$$

then

$$T : D_m^{p_0}(\mu) \longrightarrow R_m^{p_0,\infty}(\mu)$$

where

$$\|g\|_{R_m^{p_0,\infty}(\mu)} := \sup_{t>0} \frac{t^{1/p_0}g_\mu^*(t)}{(1+\log^+ t)^m},$$

and

$$\|f\|_{D_m^{p_0}(\mu)} := \|f\|_{L^{p_0}(\mu)} + \int_0^1 \left(\int_0^t f_\mu^*(s)^{p_0}ds\right)^{1/p_0}\Big(1+\log^+\frac{1}{t}\Big)^{m-1}\frac{dt}{t}.$$

It is easy to see that there is no relation between $L^{p_0,1}$ and $D_m^{p_0}$ since if f satisfies that $f_\mu^*(s) \approx \chi_{(0,1)}(s) + \frac{1}{s^{1/p_0}(1+\log_+ s)}\chi_{(1,\infty)}(s)$, then it is easy to check that $f \in D_m^{p_0}$ for every m but it is not in $L^{p_0,1}$; that is, in general, $D_m^{p_0}(\mu) \nsubseteq L^{p_0,1}(\mu)$. On the other hand, to see that $L^{p_0,1}(\mu) \nsubseteq D_m^{p_0}(\mu)$ it is enough to compute the norm of a characteristic function.

On the other hand, the endpoint estimate in Lemma 3.4 is "near" (except for the logarithmic factors) the restricted weak type estimate

$$T : L^{p_0,1}(\mu) \to L^{p_0,\infty}(\mu),$$

since

$$\|f\|_{D_m^{p_0}(ds)} \lesssim \int_0^\infty f_\mu^*(t)t^{1/p_0}\Big(1+\log^+\frac{1}{t}\Big)^m\frac{dt}{t}.$$

However, we have to mention that, under our hypothesis, we can not expect to get the restricted weak type estimate, since there are examples of operators satisfying (3.3), for which such inequality is known to be false. For instance (see [22]), the classical spherical maximal function was observed by Bourgain to be restricted weak-type at the endpoint $p = d/(d-1)$ when $d \geq 3$ but it is known to fail to be restricted weak-type at $p = 2$ when $d = 2$.

Theorem 3.5. *If* (3.2) *holds, then the Dirichlet problem is $\lhd$-solvable in the space $D_m^{p_0}(ds)$.*

Proof. As before,

$$\|M\|_{L^{p,1}(|\Phi'|)\to L^{p,\infty}(|\Phi'|)} \lesssim \||\Phi'|\|_{A_p^{\mathcal{R}}} \lesssim \frac{1}{(p-p_0)^m},$$

and hence we can apply Lemma 3.4 to obtain that

$$P^* : D_m^{p_0}(|\Phi'|) \longrightarrow R_m^{p_0,\infty}(|\Phi'|)$$

is bounded. Now, if $f \in D_m^{p_0}(|\Phi'|)$, we have that $f \in L^{p_0}(|\Phi'|)$ and hence, $\lim_{y\to\infty} \lambda_f^{|\Phi'|}(y) = 0$. Then, if we define $f_n(x) = f(x)\chi_{\{|f|\le n\}}(x)$, we have that $(f-f_n)^*_{|\Phi'|}(t) = f^*_{|\Phi'|}(t)\chi_{[0,\lambda_f^{|\Phi'|}(n)]}(t)$ and consequently $\lim_{n\to\infty}(f-f_n)^*_{|\Phi'|}(t) = 0$, and since $(f-f_n)^*_{|\Phi'|} \le f^*_{|\Phi'|}$, we have by dominated convergence that $\|f-f_n\|_{D_m^{p_0}(|\Phi'|)} \to 0$. That is, bounded functions are dense in $D_m^{p_0}(|\Phi'|)$. But if $f \in D_m^{p_0}(|\Phi'|)\cap L^\infty$, then $f \in L^p(|\Phi'|)$ with $p > p_0$ and since $|\Phi'| \in A_p$, it holds that

$$\lim_{x+it\lhd\xi}(P_t * f)(x) = f(\xi), \qquad \text{a.e. } \xi\in\mathbb{R}, \qquad \forall f \in D_m^{p_0}(|\Phi'|)\cap L^\infty.$$

Therefore, using density and the same argument than in Theorem 3.2, we obtain that

$$\lim_{x+it\lhd\xi}(P_t * f)(x) = f(\xi), \qquad \text{a.e. } \xi\in\mathbb{R}, \qquad \forall f \in D_m^{p_0}(|\Phi'|.$$

Finally, if $f : \Lambda \to \mathbb{R}$ satisfies that $f \in D_m^{p_0}(ds)$, we have by (2.7) that $f\circ\Phi \in D_m^{p_0}(|\Phi'|)$ and the result follows, as explained in Theorem 2.2. □

Corollary 3.6. *If* $1 < p_0 < 2$ *and*

$$\||\Phi'|\|_{A_p} \lesssim (p-p_0)^{-m}, \quad p_0 < p < 2,$$

the Dirichlet problem is $\lhd$*-solvable in* $D_{m/p_0}^{p_0}(ds)$.

Proof. The result follows from the previous theorem and (2.4) since

$$\||\Phi'|\|_{A_p^{\mathcal{R}}} \lesssim \||\Phi'|\|_{A_p}^{1/p} \lesssim \left(\frac{1}{p-p_0}\right)^{m/p} \approx \left(\frac{1}{p-p_0}\right)^{m/p_0}.$$

□

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