A COUNTING PROBLEM IN ERGODIC THEORY AND EXTRAPOLATION FOR ONE-SIDED WEIGHTS

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Abstract. The purpose of this paper is to prove that, given a dynamical system $(X, \mathcal{M}, \mu, \tau)$ and $0 < q < 1$, the Lorentz spaces $L^{1,q}(\mu)$ satisfy the so-called Return Times Property for the Tail contrary to what happens in the case $q = 1$. In fact, we consider a more general case than in previous papers since we work with a σ-finite measure μ and a transformation τ which is only Cesàro bounded. The proof uses the extrapolation theory of Rubio de Francia for one-sided weights. These results are of independent interest and can be applied to many other situations.

1. Introduction

Initially, let us consider $(X, \mathcal{M}, \mu, \tau)$ a finite dynamical system; that is, a finite measure space with an invertible measure preserving transformation τ on X. Then, the following result is usually referred to as Bourgain's Return Times Theorem ([7], [8]).

Theorem 1.1. *Let $(X, \mathcal{M}, \mu, \tau)$ be a finite dynamical system and let $f \in L^\infty(\mu)$. Then there exists $X_0 \subset X$ of full measure such that for all $x_0 \in X_0$, all finite dynamical systems $(Y, \mathcal{C}, \nu, S)$ and all $g \in L^1(\nu)$, the sequence of averages*

$$B_n g(y) = \frac{1}{n} \sum_{i=0}^{n-1} f(\tau^i x_0) g(S^i y)$$

converges for almost every $y \in Y$ (ν).

In short, we shall say that $L^\infty(\mu)$ satisfies RTT or simply write $L^\infty(\mu) \in RTT$.

Theorem 1.1 gives no information in the case $f \in L^1(\mu)$, however it is known that if the Return Times Theorem holds for f, then the so called

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Return Times Property for the Tail holds for this function f; that is, for all $x_0 \in X_0$, all dynamical systems $(Y, \mathcal{C}, \nu, S)$ and all $g \in L^1(\nu)$, the sequence

$$R_n g(y) = \frac{1}{n} f(\tau^n x_0) g(S^n y)$$

converges to zero for almost every $y \in Y$ (ν). In this case, we say that $f \in RTP$ and if X is a space such that $f \in RTP$ for every $f \in X$ we shall say that X satisfies the RTP or simply write $X \in RTP$. In particular,

$$X \in RTT \qquad \Longrightarrow \qquad X \in RTP.$$

Using this necessary condition and the following result, it was proved in [5] (see also [4]) that the Return Times Theorem does not hold, in general, for $L^1(\mu)$.

Theorem 1.2 ([3] Theorem 8)**.** *Let $\{c_n\}$ be a sequence of nonnegative numbers such that* $\lim_{n\to\infty} c_n = 0$*. Then, the following two statements are equivalent.*

(a) $\sup_n \frac{1}{n} \#\{k : c_k > \frac{1}{n}\} < +\infty$.
(b) *For all finite dynamical systems $(Y, \mathcal{C}, \nu, S)$ and all $g \in L^1(\nu)$, the sequence $c_n g(S^n y)$ converges to zero for almost every $y \in Y$ (ν).*

Now, given $f \in L^1(\mu)$, it is known that the sequence $c_n = \frac{f(\tau^n x)}{n}$ converges to zero a.e. x, and hence,

$$f \in RTP \qquad \Longleftrightarrow \qquad Nf(x) := \sup_{n\in\mathbb{N}} \frac{1}{n} N_{\frac{1}{n}} f(x) < +\infty \quad \text{a.e. } x,$$

where, for $\alpha > 0$,

$$N_\alpha f(x) = \#\left\{ k \geq 1 : \frac{|f(\tau^k x)|}{k} > \alpha \right\}.$$

It was proved in [5] that if the measure space is nonatomic and the transformation is ergodic then there exists $f \in L^1(\mu)$ such that $Nf(x) = +\infty$ almost everywhere, and consequently, under the mentioned hypotheses, the Return Times Property for the Tail and the Return Times Theorem do not hold for $L^1(\mu)$ functions; that is

$$L^1(\mu) \notin RTP, \qquad \text{and} \qquad L^1(\mu) \notin RTT.$$

The conclusion of our discussion is that the study of the finiteness of Nf is a key point in the Return Times theorems.

The example in [5] shows that, in general, N does not apply $L^1(\mu)$ into $L^{1,\infty}(\mu)$. However, Assani [2] proved that

$$N : L\log L(\mu) \longrightarrow L^1(\mu),$$

where

$$L\log L(\mu) = \left\{ f : \|f\|_{L\log L(\mu)} = \int_0^1 f^*_\mu(t)\Big(1 + \log^+ \frac{1}{t}\Big) dt < \infty \right\},$$

and we recall that the decreasing rearrangement of f with respect to μ is $f^*_\mu(t) = \inf\{y > 0 : \lambda^\mu_f(y) \leq t\}$, with $\lambda^\mu_f(y) = \mu(\{x : |f(x)| > y\})$ the distribution function of f with respect to μ. Hence $Nf(x) < +\infty$ a.e. x, for every $f \in L \log L(\mu)$ and therefore,

$$L \log L(\mu) \in RTP.$$

Some years later, Demeter and Quas [13] proved that

$$N : L \log \log L(\mu) \longrightarrow L^{1,\infty}(\mu),$$

and hence

$$L \log \log L(\mu) \in RTP, \tag{1.1}$$

where

$$L \log \log L(\mu) = \left\{ f : \|f\|_{L \log \log L(\mu)} = \int_0^1 f^*_\mu(t)\Big(1 + \log^+ \log^+ \frac{1}{t}\Big) dt < \infty \right\}.$$

Finally, let us just mention that it was observed in [13] that, if X is an Orlicz (or Lorentz) space strictly bigger than $L \log \log \log L(\mu)$, then $X \notin RTP$ and hence, $X \notin RTT$.

In this paper (see, for example, Theorem 2.9) we shall weaken up the previous assumptions since we shall work with σ-finite measures and τ will be an invertible measurable transformation which is Cesàro bounded in $L^1(\mu)$; that is, there exists $C > 0$ such that

$$\sup_{n \geq 1} \left\| \frac{1}{n} \sum_{i=0}^{n-1} f \circ \tau^i \right\|_{L^1(\mu)} \leq C ||f||_{L^1(\mu)}.$$

Let now $0 < p, q \leq \infty$ and let $L^{p,q}(\mu)$ be the Lorentz space defined as the space of measurable functions such that

$$\begin{aligned} ||f||_{L^{p,q}(\mu)} &= \left(q \int_0^\infty y^{q-1} \lambda^\mu_f(y)^{\frac{q}{p}} dy \right)^{1/q} \\ &= \left(\frac{q}{p} \int_0^\infty f^*_\mu(t)^q t^{\frac{q}{p}-1} dt \right)^{1/q} < \infty, \quad \text{if } q < \infty \end{aligned}$$

and if $q = \infty$,

$$||f||_{L^{p,\infty}(\mu)} = \sup_{y>0} y \lambda^\mu_f(y)^{1/p} = \sup_{t>0} t^{1/p} f^*_\mu(t) < \infty\,.$$

We recall that, if $q < r < 1$, then $L^{1,q}(\mu) \subset L^{1,r}(\mu) \subset L^1(\mu)$.

Our goal is to study the Return Times Property for the Tail and our main results (Theorem 2.9 and Corollary 2.10) will show that, for every $0 < q < 1$,

$$L^{1,q}(\mu) \in RTP. \tag{1.2}$$

In fact, we shall prove that if we consider (a bigger operator than N),

$$N^* f(x) = \sup_{\alpha > 0} \alpha N_\alpha f(x),$$

then

$$N^* : L^{1,q}(\mu) \longrightarrow L^{1,\infty}(\mu)$$

is a bounded operator and, the interesting part of the proof of this result is that it uses a new technique, developed in [10], based on the Rubio de Francia extrapolation theory. We shall need to extend this theory to the case of one-sided weights and this will be done in Section 3.

Remark 1.3. *Let μ be a non-atomic probability measure. We observe that if f is a measurable function such that*

$$f^*_\mu(t) = \frac{\chi_{(0,1)(t)}}{t(1+\log^+\frac{1}{t})(1+\log^+\log^+\frac{1}{t})^3},$$

then $f \in L\log\log L(\mu) \setminus \cup_{0<q<1} L^{1,q}(\mu)$. On the other hand, $L^{1,q}(\mu)$ is not embedded in $L\log\log L(\mu)$ since on the contrary

$$A := \sup_{f\downarrow} \frac{\int_0^\infty f(t)(1+\log^+\log^+\frac{1}{t})dt}{\left(\int_0^1 f(t)^q t^{q-1}dt\right)^{1/q}} < \infty.$$

But, it is known (see, for example, [11]*) that*

$$A = \sup_{0<r<1} \frac{\int_0^r (1+\log^+\log^+\frac{1}{t})dt}{\left(\int_0^r t^{q-1}dt\right)^{1/q}} = \infty.$$

Therefore, we see that (1.1) *and* (1.2) *are independent results since they provide non related metric spaces B such that $B \in RTP$.*

Finally, as usual, $|E|$ stands for the Lebesgue measure of the set E and if μ is the measure $d\mu = u(x)\,dx$ then λ^μ_f and f^*_μ are written λ^u_f and f^*_u. Moreover, if the measure is clearly understood we simply write λ_f and f^*. If the set X is the set of integers $\mathbb{Z}$ and the measure μ is the counting measure, the Lorentz spaces are denoted by $\ell^{p,q}$, and if the measure on the integers is given by a density $u = \{u_n\}_{n\in\mathbb{Z}}$ then, we shall write $\ell^{p,q}(u)$. In this case, for a sequence $a = \{a_n\}_{n\in\mathbb{Z}}$,

$$||a||_{\ell^{p,q}(u)} = \left(q\int_0^\infty \left(\sum_{\{n\in\mathbb{Z}:a_n>y\}} u_n\right)^{\frac{q}{p}-1} y^{q-1}dy\right)^{1/q}, \quad \text{if } q<\infty,$$

and if $q = \infty$,

$$||a||_{\ell^{p,\infty}(u)} = \sup_{y>0} y\left(\sum_{\{n\in\mathbb{Z}:a_n>y\}} u_n\right)^{1/p}.$$

A positive constant C will mean a constant independent of all important parameters. The expresion $A \lesssim B$ indicates that there exists a constant C such that $A \le CB$ and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. A counting problem for Cesàro bounded transformations

From now on, $(X, \mathcal{M}, \mu)$ will be a σ-finite measure space, $\tau : X \to X$ will be an invertible measurable transformation and we emphasize that it is easy to adapt the proof of Theorem 1.2 in [3] to show that it remains true in the case of σ-finite measures, while Theorem 1.1 may fail in this case [14].

Let us now consider the ergodic maximal operator

$$M_\tau f(x) = \sup_{n\geq 1} A_n|f|(x), \qquad A_n f = \frac{1}{n}\sum_{i=0}^{n-1} f\circ \tau^i.$$

We need the following result that can be found in [23] and in [27].

Theorem 2.1. *Let $1 \leq p < +\infty$ and let us assume that τ is Cesàro bounded in $L^p(\mu)$; that is,*

$$\sup_{n\geq 1} \|A_n f\|_{L^p(\mu)} \lesssim ||f||_{L^p(\mu)}, \qquad \forall f \in L^p(\mu).$$

Then:

(a) *If $p = 1$*

$$\|M_\tau f\|_{L^{1,\infty}(\mu)} \lesssim ||f||_{L^1(\mu)}, \qquad \forall f \in L^1(\mu).$$

(b) *If $1 < p < +\infty$,*

$$||M_\tau f||_{L^p(\mu)} \leq C_p||f||_{L^p(\mu)}, \qquad \forall f \in L^p(\mu).$$

Definition 2.2. *Let $1 \leq p < \infty$. An operator T is said to be of restricted weak type (p, p) if there exists a constant $C > 0$ such that, for all measurable sets E,*

$$||T\chi_E||_{L^{p,\infty}(\mu)} \leq C\mu(E)^{1/p}.$$

The least of all possible constants C is denoted by $\|T\|_{p,rest}$.

We observe that the operator N^* is closely related to the ergodic maximal operator since, if A is a measurable set, then $N^*(\chi_A) = M_\tau(\chi_A)$. Hence, using Theorem 2.1, we immediately have the following result:

Corollary 2.3. *Under the hypotheses of Theorem 2.1, we have that N^* is of restricted weak type (p, p).*

Theorem 2.4. *Let $1 < p < +\infty$ and let us assume that τ is Cesàro bounded in $L^p(\mu)$. Then*

$$N^* : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\mu)$$

is bounded with

$$||N^*||_{L^{p,1}(\mu)\to L^{p,\infty}(\mu)} \leq \frac{4p\|N^*\|_{p,rest}}{p-1}.$$

Proof. Let $f \in L^{p,1}(\mu)$. For each integer number i, define $E_i = \{x : 2^{i-1} < |f(x)| \leq 2^i\}$, and $f_i = f\chi_{E_i}$. Using that N^* is sublinear on disjointly supported functions and monotone, we have

$$N^* f \leq \sum_{i=-\infty}^{\infty} N^* f_i \leq \sum_{i=-\infty}^{\infty} 2^i N^* \chi_{E_i}.$$

Since $p > 1$ and N^* is of restricted weak type (p,p),

$$\begin{aligned}
\|N^* f\|_{L^{p,\infty}(\mu)} &\leq \frac{p}{p-1} \sum_{i=-\infty}^{\infty} 2^i \|N^*\chi_{E_i}\|_{L^{p,\infty}(\mu)} \leq \frac{p\|N^*\|_{p,rest}}{p-1} \sum_{i=-\infty}^{\infty} 2^i \mu(E_i)^{\frac{1}{p}} \\
&\leq \frac{4p}{p-1} \|N^*\|_{p,rest} \sum_{i=-\infty}^{\infty} \int_{2^{i-2}}^{2^{i-1}} \mu(\{x : |f(x)| > t\})^{\frac{1}{p}}\, dt \\
&= \frac{4p\|N^*\|_{p,rest}}{p-1} \|f\|_{L^{p,1}(\mu)},
\end{aligned}$$

and the result follows. □

Remark 2.5. *Notice that if $1 < p < \infty$ and τ is an invertible measurable transformation which is Cesàro bounded in $L^p(\mu)$ then it is Cesàro bounded in $L^{p-\varepsilon}(\mu)$ for some ε, $0 < \varepsilon < p-1$ (see* [23] *and* [27]*).*

Using Remark 2.5, Theorem 2.4 and Marcinkiewicz's interpolation theorem (adapted to sublinear operators on disjointly supported functions) one can easily prove the boundedness on $L^p(\mu)$:

Theorem 2.6. *Let $1 < p < +\infty$ and let us assume that τ is Cesàro bounded in $L^p(\mu)$. Then,*

$$N^* : L^p(\mu) \longrightarrow L^p(\mu)$$

is bounded.

As mentioned in the introduction, we already know that N^* does not apply $L^1(\mu)$ into $L^{1,\infty}(\mu)$. However, we can prove some boundedness result on $L^{1,q}(\mu)$ for every $0 < q < 1$. To this end, we need to first assume the following Claim and then prove a previous proposition.

Let us consider the continuous version of the operator N^* defined by

$$N^{*,c} f(x) = \sup_{\alpha > 0} \alpha \left| \left\{ y > 0 : \frac{|f(x+y)|}{y} > \alpha \right\} \right|,$$

and let us recall the definition of the one-sided weights $u \in A_1^+$: a locally integrable positive function $u \in A_1^+$ if there exists $C > 0$ such that $M^- u(x) \leq C u(x)$, a.e. x, where

$$M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)| dy,$$

and $||w||_{A_1^+}$ will denote the infimum of such constants $C > 0$.

CLAIM: *Let* $0 < q < 1$ *and let* $u \in A_1^+$. *Then*

$$N^{*,c} : L^{1,q}(u) \longrightarrow L^{1,\infty}(u) \tag{2.1}$$

is bounded with a constant less than or equal to $C_q \|u\|_{A_1^+}$, *for some constant* $C_q > 0$ *depending only on* q.

We should mention here that the whole Section 3 will be devoted to the proof of this claim.

Let us now take the dynamical system $(X, \mathcal{M}, \mu, \tau)$ where $X = \mathbb{Z}$ is the set of the integers, $\mathcal{M} = \mathcal{P}(X)$, μ is the counting measure ($\mu(E) = \#(E)$) and $\tau(i) = i + 1$. The counting function N^* associated to this dynamical system will be denoted by $N^{*,d}$; that is, if f is a function defined on the integers

$$N^{*,d} f(i) = \sup_{\alpha>0} \alpha \,\#\left\{k \geq 1 : \frac{|f(i+k)|}{k} > \alpha\right\}.$$

A weight $w \in A_1^+(\mathbb{Z})$ is a nonnegative function on the integers such that

$$m^- w(j) = \sup_{n\geq 1} \frac{1}{n} \sum_{i=0}^{n-1} w(j-i) \leq C w(j) \tag{2.2}$$

for all $j \in \mathbb{Z}$, where C is a positive constant independent of j and

$$||w||_{A_1^+(\mathbb{Z})} = \inf\{C > 0;\ (2.2) \text{ holds}\}.$$

Remark 2.7. *Let* w *be a nonnegative function on* $\mathbb{Z}$ *and let us define* $W : \mathbb{R} \to \mathbb{R}$ *by* $W(x) = w([x])$, *where* $[x]$ *is the integer part of* x. *Then, it is easy to see that,* $w \in A_1^+(\mathbb{Z})$ *if and only if* $W \in A_1^+$ *and* $||w||_{A_1^+(\mathbb{Z})} \approx ||W||_{A_1^+}$.

Now, using the CLAIM and a discretization argument we can prove the following result.

Proposition 2.8. *Let* $0 < q < 1$ *and let* $w \in A_1^+(\mathbb{Z})$. *Then*

$$N^{*,d} : \ell^{1,q}(w) \longrightarrow \ell^{1,\infty}(w)$$

is bounded with constant $C_q \|w\|_{A_1^+(\mathbb{Z})}$, *for some constant* $C_q > 0$ *depending only on* q.

Proof. Let $f : \mathbb{Z} \to \mathbb{R}^+$ and let us define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = f([x])$. We notice that if $\alpha < \frac{f(i+k)}{k}$, $x \in (i, i+1)$ and $z \in (i+k, i+k+1)$ then

$$\begin{aligned}\alpha < \frac{f(i+k)}{k} &= \frac{F(z)}{k} = \frac{F(x+z-x)}{z-x}\frac{z-x}{k} \\ &\leq \frac{F(x+z-x)}{z-x}\frac{k+1}{k} \leq 2\frac{F(x+z-x)}{z-x}.\end{aligned}$$

Consequently,

$$\bigcup_{\left\{k\geq 1:\frac{f(i+k)}{k}>\alpha\right\}} (i+k, i+k+1) \subset \left\{z > x : \frac{F(x+z-x)}{z-x} > \frac{\alpha}{2}\right\}.$$

That implies

$$\#\left\{k \geq 1 : \frac{f(i+k)}{k} > \alpha\right\} \leq \left|\left\{y > 0 : \frac{F(x+y)}{y} > \frac{\alpha}{2}\right\}\right|,$$

and thus, $N^{*,d}f(i) \leq 2N^{*,c}F(x)$ for all $x \in (i, i+1)$. It follows that

$$||N^{*,d}f||_{\ell^{1,\infty}(w)} \leq 2||N^{*,c}F||_{L^{1,\infty}(W)}.$$

By Remark 2.7 and the CLAIM, we obtain

$$||N^{*,d}f||_{\ell^{1,\infty}(w)} \lesssim C_q||W||_{A_1^+}||F||_{L^{1,q}(W)} \approx C_q||w||_{A_1^+(\mathbb{Z})}||f||_{\ell^{1,q}(w)},$$

as we wanted to prove. □

We are now ready to formulate and prove our first main result, via a classical transference argument:

Theorem 2.9. *Let* $0 < q < 1$ *and let us assume that* τ *is Cesàro bounded in* $L^1(\mu)$*. Then*

$$N^* : L^{1,q}(\mu) \longrightarrow L^{1,\infty}(\mu)$$

is bounded.

Proof. It suffices to prove the theorem for nonnegative measurable functions. The assumption on τ implies that τ is a nonsingular transformation (it preserves the sets of measure zero). By the Radon-Nikodym theorem (see [27]) there exists a family of nonnegative measurable functions $\{J_i(x)\}_{i\in\mathbb{Z}}$ such that $J_{i+j}(x) = J_i(\tau^j x)J_j(x)$ and for all nonnegative measurable functions

$$\int_X f\, d\mu = \int_X f(\tau^i(x))J_i(x)\, d\mu(x).$$

Furthermore, it is known [27] that

$$h_x(i) = J_i(x) \in A_1^+(\mathbb{Z}), \quad \text{a.e. } x, \text{ and } ||h_x||_{A_1^+(\mathbb{Z})} \leq \sup_n ||A_n||_{L^1(\mu)}. \tag{2.3}$$

Let us fix a natural number L and consider

$$(N_\alpha)_L f(x) = \#\left\{1 \leq k \leq L : \frac{|f(\tau^k x)|}{k} > \alpha\right\}$$

and the truncated operator $N_L^* f(x) = \sup_{\alpha>0}\alpha(N_\alpha)_L f(x)$. Let us fix a nonnegative measurable function f. For any $x \in X$, set f^x the function defined on $\mathbb{Z}$ by $f^x(i) = f(\tau^i x)$. Let $n \in \mathbb{N}$ and $\lambda > 0$. If $O_{\lambda,L} = \{x : N_L^* f(x) > \lambda\}$ then

$$\lambda\mu(O_{\lambda,L}) = \frac{1}{n+1}\int_X \lambda\sum_{i=0}^n \chi_{O_{\lambda,L}}(\tau^i x)h_x(i)\, d\mu(x).$$

It is easy to see that if $\chi_{O_{\lambda,L}}(\tau^i x) = 1$ $(0 \le i \le n)$, then $N^{*,d}(f^x \chi_{[0,n+L]})(i) > \lambda$. Then for all x,

$$\lambda \sum_{i=0}^{n} \chi_{O_{\lambda,L}}(\tau^i x) h_x(i) \le \lambda \sum_{\{i : N^{*,d}(f^x \chi_{[0,n+L]})(i) > \lambda\}} h_x(i)$$
$$\le ||N^{*,d}(f^x \chi_{[0,n+L]})||_{\ell^{1,\infty}(h_x)}.$$

Since, by (2.3), for a.e. $x \in X$, the functions $h_x \in A_1^+(\mathbb{Z})$ with a uniform constant, Proposition 2.8 asserts that there exists $C_q > 0$ such that

$$||N^{*,d}(f^x \chi_{[0,n+L]})||_{\ell^{1,\infty}(h_x)} \le C_q \|f^x \chi_{[0,n+L]}\|_{\ell^{1,q}(h_x)} \quad \text{a.e. } x.$$

Therefore,

$$\lambda \mu(O_{\lambda,L}) \le \frac{C_q}{n+1} \int_X \|f^x \chi_{[0,n+L]}\|_{\ell^{1,q}(h_x)} d\mu(x)$$
$$= \frac{C_q}{n+1} \int_X \left(q \int_0^\infty \left(t \sum_{\{i \in [0,n+L] : f^x(i)\} > t} h_x(i) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} d\mu(x).$$

Applying Minkowski's integral inequality

$$\lambda \mu(O_{\lambda,L}) \le \frac{C_q}{n+1} q^{\frac{1}{q}} \left(\int_0^\infty t^q \left(\int_X \sum_{\{i \in [0,n+L] : f^x(i) > t\}} h_x(i)\, d\mu(x) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$
$$= \frac{C_q}{n+1} q^{\frac{1}{q}} \left(\int_0^\infty t^q \left(\sum_{i=0}^{n+L} \int_{\{x : f(\tau^i x) > t\}} h_x(i)\, d\mu(x) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$
$$= C_q q^{\frac{1}{q}} \frac{n+L+1}{n+1} \left(\int_0^\infty (t \mu\{x : f(x) > t\})^q \frac{dt}{t} \right)^{\frac{1}{q}}$$
$$= C_q \frac{n+L+1}{n+1} ||f||_{L^{1,q}(\mu)}.$$

Letting n and then L tend to $+\infty$ we obtain the desired result. □

The final result of this section is our second main result:

Corollary 2.10. *Under the hypotheses of Theorem 2.9,*

$$L^{1,q}(\mu) \in RTP.$$

Proof. The proof follows easily since for every $f \in L^{1,q}(\mu)$, $0 < q < 1$, it holds that $\lim_{n\to\infty} \frac{f(\tau^n x)}{n} = 0$, a.e. x, because the averages converge a.e. (see [23] and [27]). □

With a similar argument we also obtain the following result.

Corollary 2.11. *Under the hypotheses of Theorem 2.6,*

$$L^p(\mu) \in RTP.$$

3. Extrapolation for one-sided weights

The purpose of this section is to give the proof of the CLAIM. However, the results obtained here are of independent interest and can be applied to many other situations. The main idea is to obtain (2.1) for every $u \in A_1^+$, by using the Rubio de Francia extrapolation argument recently developed in [10] adapted to the case of one-sided weights. At this point, we want to emphasize that we shall only give the proofs of those results which do not easily follow from the two-sided case.

3.1. **One-sided weights.** As mentioned in the previous section, by a weight we understand a locally integrable function $w \geq 0$. The good weights for the classical Hardy-Littlewood maximal operator, M, are the weights in the classes A_p of Muckenhoupt [25]. One-sided weights are the good weights for one-sided operators like the one-sided Hardy-Littlewood maximal functions, defined in $\mathbb{R}$ for locally integrable functions f by

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f| \quad \text{and} \quad M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f|.$$

In general, a one-sided operator T^+ (respectively T^-) in $\mathbb{R}$ is an operator such that the value of $T^+f(x)$ (respectively $T^-f(x)$) depends only on the values of f in $[x, \infty)$ (respectively $(-\infty, x]$) (see some examples of one-sided operators in [1], [18], [19] [16], [6], [29], [20]). Many classical operators in Real Analysis are one-sided versions for which the class of weights is wider than the one of Muckenhoupt.

The one-sided A_p weights, $1 \leq p < \infty$, were introduced by E. Sawyer [28] and they are defined as follows: We say that $w \in A_p^+$, $p > 1$, if

$$||w||_{A_p^+} = \sup_{h>0} \left(\frac{1}{h} \int_{x-h}^{x} w \right) \left(\frac{1}{h} \int_x^{x+h} w^{-\frac{1}{p-1}} \right)^{p-1} < \infty,$$

where A_1^+ is defined as in Section 2. The following results about these weights can be found in [28] and [22].

(1) The operator M^+ is of weak type $(1, 1)$ with respect to w if and only if $w \in A_1^+$.
(2) The operator M^+ is a bounded operator on $L^p(w)$, $p > 1$, if and only if, $w \in A_p^+$.
(3) If $M^-f < \infty$ a.e. then $(M^-f)^\delta \in A_1^+$ for every $0 < \delta < 1$ and $||(M^-f)^\delta||_{A_1^+} \leq C\frac{1}{1-\delta}$.
(4) Factorization: $w \in A_p^+$ if an only if $w = u_0 u_1^{1-p}$ for some $u_0 \in A_1^+$ and $u_1 \in A_1^-$.

Of course there are similar definitions and theorems for A_p^-, that are obtained reversing the orientation of $\mathbb{R}$. All the results that we shall state in this section have the corresponding result reversing the orientation of the real line.

Remark 3.1. *i) Unlike the two-sided case, f not identically zero does not imply $M^+f > 0$ a.e. and therefore the one-sided weights can be zero in a set of positive measure. Here we are adopting the usual conventions $\infty \cdot t = t \cdot \infty = \infty$, for $0 < t \leq \infty$, $0 \cdot \infty = \infty \cdot 0 = 0$, $\infty^{-1} = 0$ and $0^{-1} = \infty$.*
ii) From the definitions we have the following: if $w \in A_p^+$ then there exist a and b, $-\infty \leq a \leq b \leq \infty$ such that $w = 0$ in $(-\infty, a)$, $w = \infty$ in (b, ∞), $0 < w < \infty$ in (a, b), $w \in L^1_{loc}(a, b)$ and, if $1 < p < \infty$and $p' = \frac{p}{p-1}$ is the dual exponent of p, $w^{1-p'} \in L^1_{loc}(a, b)$ (see [15]). Then, when we are working with one-sided weights, we can assume without loss of generality that $(a, b) = \mathbb{R}$.

There is a great amount of works dealing with one-sided weights and one-sided operators, extending the results for the standard cases to the one-sided case. Most of the time this involves significant technical difficulties.

In this section we extend the results in [10] to the setting of one-sided weights. We shall use them to prove the CLAIM in Section 2. Other extrapolation results for one-sided weights can be found in [12], [21], [17], [9].

3.2. **Restricted weak-type extrapolation.**

Definition 3.2. *Let $1 \leq p < \infty$. We will say that a weight $w \in A_p^{\mathcal{R},+}$ if there exists a constant $C > 0$ such that, for any three numbers $a < b < c$ and any measurable set $E \subset (b, c)$,*

$$w((a,b)) \left(\frac{|E|}{c-a} \right)^p \leq Cw(E).$$

And we will denote by $||w||_{A_p^{\mathcal{R},+}}$ the infimum of these constants.

It is easy to see that it suffices to check the condition only for $a < b < c$ such that $b - a = c - b$ (the constant appearing is $2^p C$). The following result gives a characterization of the class $A_p^{\mathcal{R},+}$.

Lemma 3.3. *Let $1 < p < +\infty$. The weight $w \in A_p^{\mathcal{R},+}$ if and only if*

$$\sup_{a<b<c} \frac{||\chi_{(a,b)}||_{L^{p,1}(w)} ||w^{-1}\chi_{(b,c)}||_{L^{p',\infty}(w)}}{c-a} < +\infty.$$

Proof. See Lemma 3 in [26]. □

Theorem 3.4. [24] *Let $1 \leq p < \infty$ and let u be a weight. Then*

$$M^+ : L^{p,1}(u) \to L^{p,\infty}(u)$$

is bounded if, and only if, $u \in A_p^{\mathcal{R},+}$. Furthermore, if $1 < p < \infty$,

$$(p-1)||M^+||_{L^{p,1}(u)\to L^{p,\infty}(u)} \lesssim ||u||_{A_p^{\mathcal{R},+}} \leq ||M^+||_{L^{p,1}(u)\to L^{p,\infty}(u)},$$

and, if $p = 1$,

$$||M^+||_{L^1(u)\to L^{1,\infty}(u)} \approx ||u||_{A_1^+}.$$

Following the same steps as in the two-sided case, it is easy to see that the following result also holds:

Proposition 3.5. *For every* $\varepsilon > 0$, $A_p^+ \subset A_p^{\mathcal{R},+} \subset A_{p+\varepsilon}^+$ *and* $||u||_{A_p^{\mathcal{R},+}} \leq ||u||_{A_p^+}^{1/p}$.

3.3. Construction of $A_p^{\mathcal{R},+}$ weights.

Lemma 3.6. *Suppose* $M^+f(x) < \infty$ *a.e. and* f *is not identically* 0 *a.e. Let* $a < c < d$ *such that* $d - c = c - a$. *If* $g = f\chi_{(d,\infty)}$ *then there exists* K, $0 < K < +\infty$ *such that* $\frac{K}{2} \leq M^+g(x) \leq K$ *for all* $x \in (a, c)$.

Proof. Let $f \geq 0$. Let $x, y \in (a, c)$, $x \leq y$ and $h > 0$. Then

$$\frac{1}{h+d-x}\int_d^{d+h} f \leq \frac{1}{h+d-y}\int_d^{d+h} f \leq M^+g(y).$$

It follows that $M^+g(x) \leq M^+g(y)$. In analogous way

$$\begin{aligned}\frac{1}{h+d-y}\int_d^{d+h} f &\leq \frac{h+d-x}{h+d-y}\cdot\frac{1}{h+d-x}\int_d^{d+h} f \leq \frac{h+d-x}{h+d-y}M^+g(x)\\ &= \Big(1+\frac{y-x}{h+d-y}\Big)M^+g(x) \leq \Big(1+\frac{c-a}{d-c}\Big)M^+g(x)\\ &= 2M^+g(x).\end{aligned}$$

Therefore $M^+g(y) \leq 2M^+g(x)$. Taking $K = \sup_{x\in(a,c)} M^+g(x)$, we have the result. □

The following results (Theorem 3.7 and Corollary 3.8) will be proved in a completely different way that in the two-sided case.

Theorem 3.7. *Let* $1 < p < +\infty$. *Suppose that* $(M^+f)(x) < \infty$ *a.e. and* f *not identically* 0 *(a.e.). Then* $w = (M^+f)^{1-p} \in A_p^{\mathcal{R},+}$, *with constant independent of* f.

Proof. By Lemma 3.3, it suffices to prove that there exists $C > 0$ such that for all $a < b < c$ and for all $t > 0$

$$\Big(\int_a^b w\Big)^{1/p} t\Big(\int_{\{x\in(b,c):w^{-1}(x)>t\}} w\Big)^{1/p'} \leq C(c-a).$$

Let $a < b < c$ and let d such that $d - c = c - a$. Consider K and g like in Lemma 3.6. Then $w \leq (M^+g)^{1-p}$ since $g \leq f$.

If $0 < t \le (2K)^{p-1}$ then

$$\begin{aligned}
&\left(\int_a^b w\right)^{1/p} t \left(\int_{\{x\in(b,c):w^{-1}(x)>t\}} w\right)^{1/p'} \\
&\le t\left(\int_a^b (M^+g)^{1-p}\right)^{1/p}\left(\int_{x\in(b,c)} (M^+g)^{1-p}\right)^{1/p'} \\
&\le t\left(\int_a^b (K/2)^{1-p}\right)^{1/p}\left(\int_{x\in(b,c)} (K/2)^{1-p}\right)^{1/p'} \\
&\le t(K/2)^{1-p}(b-a)^{1/p}(c-b)^{1/p'} \\
&\le (2K)^{p-1}(K/2)^{(1-p)}(c-a) = 4^{p-1}(c-a).
\end{aligned}$$

If $t > (2K)^{p-1}$ then for all $x \in (b,c)$,

$$\begin{aligned}
M^+f(x) \le M^+(f\chi_{(b,d)})(x) + M^+g(x) &\le M^+(f\chi_{(b,d)})(x) + K \\
&< M^+(f\chi_{(b,d)})(x) + \frac{t^{1/(p-1)}}{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\{x\in(b,c): w^{-1}(x) > t\} &= \left\{x\in(b,c): M^+f(x) > t^{1/(p-1)}\right\} \\
&\subset \left\{x\in(b,c): (M^+(f\chi_{(b,d)})(x))^{p-1} > \frac{t}{2^{p-1}}\right\}.
\end{aligned}$$

Applying that M^+ is of weak type $(1,1)$ respect to the Lebesgue's measure,

$$\begin{aligned}
&t\left(\int_{\{x\in(b,c):w^{-1}(x)>t\}} w\right)^{1/p'} \le t\left(\int_{\left\{x\in(b,c):(M^+(f\chi_{(b,d)})(x))^{p-1}>\frac{t}{2^{p-1}}\right\}} w\right)^{1/p'} \\
&\le t^{1/p}\left(t\int_{\left\{x\in(b,c):(M^+(f\chi_{(b,d)})(x))^{p-1}>\frac{t}{2^{p-1}}\right\}} (M^+(f\chi_{(b,d)}))^{1-p}\right)^{1/p'} \\
&\le t^{1/p}\left(\left|\left\{x\in(b,c): M^+(f\chi_{(b,d)})(x) > \frac{t^{1/(p-1)}}{2}\right\}\right| 2^{p-1}\right)^{1/p'} \\
&= 2^{(p-1)/p'}t^{1/p}\left(\frac{2}{t^{1/(p-1)}}\int_b^d f\right)^{1/p'} = 2^{p-1}\left(\int_b^d f\right)^{1/p'}.
\end{aligned}$$

Consequently,

$$\left(\int_a^b w\right)^{1/p} t\left(\int_{\{x\in(b,c):w^{-1}(x)>t\}} w\right)^{1/p'} \leq \left(\int_a^b w\right)^{1/p} 2^{p-1}\left(\int_b^d f\right)^{1/p'}$$
$$\leq 2^{p-1}\left(\int_a^b w(M^+f)^{p-1}\right)^{1/p} (d-a)^{1/p'}$$
$$= 2^{p-1}(b-a)^{1/p}(2(c-a))^{1/p'} \leq 2^{p-1/p}(c-a).$$

□

Corollary 3.8. *Suppose that* $(M^+f)(x) < \infty$ *a.e. and* f *not identically* 0 *(a.e.). Let* $1 \leq p < \infty$*. If* $u \in A_1^+$ *then* $w = (M^+f)^{1-p}u \in A_p^{\mathcal{R},+}$ *and*

$$||w||_{A_p^{\mathcal{R},+}} \lesssim ||u||_{A_1^+}^{1/p}.$$

Proof. Let $a < b < c$ and let $E \subset (b,c)$ be a measurable set. For $x \in (a,b)$, let us consider the following decreasing sequence: $z_0 = c$ and for $n \in \mathbb{N}$, $z_{n+1} = \frac{x+z_n}{2}$.

For $\alpha \in (0,1)$, $(M^+f)^\alpha \in A_1^-$, with constant depending only on α. Then, for every $n \in \mathbb{N}$,

$$\frac{1}{z_{n+1}-x}\int_x^{z_{n+1}} (M^+f)^\alpha \leq M^+(M^+f)^\alpha(x) \leq C(M^+f)^\alpha(x). \tag{3.1}$$

Now, for every $q > 1$, we get from Hölder's inequality that

$$(z_{n+1}-x)^q \leq \left(\int_x^{z_{n+1}} (M^+f)^\alpha\right)\left(\int_x^{z_{n+1}} (M^+f)^{-\alpha/(q-1)}\right)^{q-1}.$$

Therefore

$$\frac{z_{n+1}-x}{\int_x^{z_{n+1}}(M^+f)^\alpha} \leq \left(\frac{1}{z_{n+1}-x}\int_x^{z_{n+1}} (M^+f)^{-\alpha/(q-1)}\right)^{q-1}.$$

Let us take $q = 1 + \frac{\alpha}{p-1}$. Then, the last inequality and (3.1) give

$$C(M^+f(x))^{-\alpha} \leq \left(\frac{1}{z_{n+1}-x}\int_x^{z_{n+1}} (M^+f)^{1-p}\right)^{\frac{\alpha}{p-1}}.$$

Raising to the power $\frac{p-1}{\alpha p} > 0$, using that $(M^+f)^{1-p} \in A_p^{\mathcal{R},+}$ with $x < z_{n+1} < z_n$ and $E_n = E \cap (z_{n+1}, z_n)$ and taking into account that $z_n - x = 2(z_n - z_{n+1})$ and $z_{n+1} - x = z_n - z_{n+1}$, we get

$$(M^+f(x))^{\frac{1-p}{p}}|E_n| \leq 2C(z_n - z_{n+1})^{(p-1)/p}\left(\int_{E_n} (M^+f)^{1-p}\right)^{1/p}.$$

Summing up in $n \in \mathbb{N}$ and using Hölder inequality with exponents (p', p),

$$\begin{aligned}(M^+f(x))^{\frac{1-p}{p}}|E| &= (M^+f(x))^{\frac{1-p}{p}}\sum_{n=0}^{\infty}|E_n| \\ &\le C\left(\sum_{n=0}^{\infty}(z_n - z_{n+1})\right)^{(p-1)/p}\left(\sum_{n=0}^{\infty}\int_{E_n}(M^+f)^{1-p}\right)^{1/p} \\ &= C(c-x)^{(p-1)/p}\left(\int_E (M^+f)^{1-p}\right)^{1/p}.\end{aligned}$$

Therefore, we have obtained that

$$(M^+f(x))^{1-p}|E|^p \le C(c-a)^{p-1}\int_E (M^+f)^{1-p}, \tag{3.2}$$

for almost every $x \in (a,b)$. Now, since $u \in A_1^+$, for almost every $y \in E$ we have that

$$\frac{1}{c-a}\int_a^b u(x)dx \le \frac{1}{y-a}\int_a^y u(x)dx \le M^-u(y) \le ||u||_{A_1^+}u(y).$$

Then, multiplying in (3.2) by $u(x)$ and integrating in (a,b), we get

$$\begin{aligned}&|E|^p\int_a^b (M^+f(x))^{1-p}u(x)dx \\ &\quad\le C(c-a)^p\frac{1}{c-a}\int_a^b u(x)dx\int_E (M^+f)^{1-p}(y)dy \\ &\quad\lesssim ||u||_{A_1^+}(c-a)^p\int_E (M^+f)^{1-p}(y)u(y)dy\,.\end{aligned}$$

That is $w = (M^+f)^{1-p}u \in A_p^{\mathcal{R},+}$. □

Now, we are going to define the class of weights that we shall use to extrapolate.

Definition 3.9. *Let $1 \le p < \infty$. We will say that a weight w belongs to $\widehat{A}_p^+$ if there exist $f \in L^1_{loc}$ and $u \in A_1^+$ such that $w = (M^+f)^{1-p}u$, with $||w||_{\widehat{A}_p^+} = \inf ||u||_{A_1^+}^{1/p}$.*

By Corollary 3.8, $\widehat{A}_p^+ \subset A_p^{\mathcal{R},+}$ and $||w||_{A_p^{\mathcal{R},+}} \lesssim ||w||_{\widehat{A}_p^+}$.

The following distribution inequality will be used in the proof of the first extrapolation result. Its proof follows exactly the same pattern as Proposition 2.10 in [10], and we omit it.

Proposition 3.10. *Let u be a weight, f and g two positive functions, $\gamma > 0$ and $1 \le p < p_0$. Then*

$$\lambda_g^u(y) \le \lambda_{M^+f}^u(\gamma y) + \gamma^{p_0-p}\frac{y^{p_0}}{y^p}\int_{\{x:g(x)>y\}}(M^+f)^{p-p_0}(x)u(x)dx.$$

Now we state the extrapolation result, which follows also the same pattern as the two-sided case.

Theorem 3.11. *Let T be a sublinear operator satisfying that, for some $p_0 > 1$ and every $v \in \widehat{A}^+_{p_0}$,*

$$||Tf||_{L^{p_0,\infty}(v)} \le \varphi_{p_0}(||v||_{\widehat{A}^+_{p_0}})||f||_{L^{p_0,1}(v)}\,,$$

with φ_{p_0} an increasing function on $(0,\infty)$. Then, for every $1 \le p < p_0$, and every $v \in \widehat{A}^+_p$,

$$||Tf||_{L^{p,\infty}(v)} \le C||v||^{1-p/p_0}_{\widehat{A}^+_p}\varphi_{p_0}(C||v||^{p/p_0}_{\widehat{A}^+_p})||f||_{L^{p,p/p_0}(v)}\,.$$

In particular, T is of restricted weak-type (p,p) with respect to v.

Proof of the CLAIM

Using Theorem 3.4 and the fact that $N^{*,c}\chi_E = M^+\chi_E$, we easily have, arguing as in Theorem 2.4, that for every $1 < p < +\infty$ and every $u \in A^{\mathcal{R},+}_p$,

$$N^{*,c} : L^{p,1}(u) \longrightarrow L^{p,\infty}(u)$$

is bounded with $||N^{*,c}||_{L^{p,1}(u)\to L^{p,\infty}(u)} \lesssim \frac{||u||_{A^{\mathcal{R},+}_p}}{p-1}$, and hence, by Theorem 3.11, we finally obtain (2.1).

Remark 3.12. *Under the same hypothesis of Theorem 3.11 in the case $p = 1$, we also get that it is possible to extrapolate up to a space quite near to $L^1(u)$, for $u \in A^+_1$. Namely, for every $\varepsilon > 0$,*

$$||Tf||_{L(\log L)^\varepsilon(u)} \le C||f||_{L^{1,\infty}_{loc}(u)}\,,$$

with constant $C \lesssim \frac{1}{\varepsilon}||u||^{1-1/p_0}_{A^+_1}\varphi_{p_0}(||u||^{1/p_0}_{A^+_1})$, where the spaces $L(\log L)^\varepsilon(u)$ and $L^{1,\infty}_{loc}(u)$ are defined by the conditions

$$||f||_{L(\log L)^\varepsilon(u)} = \int_0^\infty f^*_u(t)\left(1+\log^+\frac{1}{t}\right)^\varepsilon dt < \infty,$$

$$||f||_{L^{1,\infty}_{loc}(u)} = \sup_{0<t\le 1} t f^*_u(t) < \infty,$$

respectively. The proof of this fact follows the same pattern as in [10].

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