

# ENDPOINT ESTIMATES FOR RUBIO DE FRANCIA OPERATORS

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ABSTRACT. The extrapolation theory of Rubio de Francia provides a tool to obtain  $A_p$ -weighted estimates on  $L^p$  spaces for every  $1 < p < \infty$ , starting from information at a single  $1 < p_0 < \infty$ . However, the endpoint case  $p = 1$  cannot be reached in general. Classical extrapolation arguments in the sense of Yano can be added to this setting to deduce results close to  $L^1$  without weights. In this paper, we present different approaches that produce endpoint estimates with respect to the whole  $A_1$  class. We give applications to the Carleson operator and maximally modulated singular integrals among others.

## 1. INTRODUCTION AND MOTIVATION

In 1984, J. L. Rubio de Francia [36] presented one of the most important extrapolation results in the context of weighted  $L^p$  spaces. Recall that, given a measure space  $(\Omega, \mu)$  and  $1 \leq p < \infty$ , we define  $L^p(\mu)$  as the set of  $\mu$ -measurable functions satisfying

$$\|f\|_{L^p(\mu)} = \left( \int_{\Omega} |f(x)|^p d\mu(x) \right)^{1/p} < \infty.$$

When  $\Omega = \mathbb{R}^n$  and the measure  $d\mu = w dx$  is given by a non-negative, locally integrable function  $w$  (that we call weight), then we write  $L^p(w)$ . Rubio de Francia's extrapolation theorem can be stated as follows:

**Theorem 1.1.** *Given a sublinear operator  $T$ , if for some  $1 \leq p_0 < \infty$  and every  $w \in A_{p_0}$ ,*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

*is bounded, then, for every  $1 < p < \infty$  and every  $w \in A_p$ ,*

$$T : L^p(w) \longrightarrow L^p(w)$$

*is also bounded.*

The classes of weights appearing in Theorem 1.1 are the so-called  $A_p$  weights, introduced by B. Muckenhoupt [33] in 1972. For  $1 < p < \infty$ , we say that  $w \in A_p$  whenever

$$\|w\|_{A_p} = \sup_Q \frac{w(Q)}{|Q|} \left( \frac{w^{1-p'}(Q)}{|Q|} \right)^{p-1} < \infty,$$

and  $w \in A_1$  if

$$Mu(x) \leq Cu(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

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with  $\|u\|_{A_1}$  being the least constant  $C \geq 1$  that can be taken in such an inequality. The operator  $M$  involved is the Hardy-Littlewood maximal operator, defined on locally integrable functions  $f$  by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes  $Q \subseteq \mathbb{R}^n$  containing  $x$ . In fact, as shown by Muckenhoupt in the same paper, these  $A_p$  classes characterize the  $L^p$  boundedness of  $M$ . More precisely, when  $1 < p < \infty$ ,

$$M : L^p(w) \longrightarrow L^p(w)$$

is bounded if and only if  $w \in A_p$ , and

$$M : L^1(u) \longrightarrow L^{1,\infty}(u)$$

is bounded if and only if  $u \in A_1$ .

Since 1984, Theorem 1.1 has been improved in different directions (see [16, 17, 21] for further details). For instance, it can be shown that the result is still true for general operators (not necessarily sublinear), or if we change the strong-type estimates by weak-type ones. Also, the following quantitative version of Theorem 1.1 was given in [20] (see also [21]):

**Theorem 1.2.** *Given an operator  $T$ , if for some  $1 < p_0 < \infty$  and every  $w \in A_{p_0}$ ,*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

*is bounded with constant  $\varphi(\|w\|_{A_{p_0}})$ , with  $\varphi$  an increasing function on  $(0, \infty)$ , then, for every  $1 < p < p_0$  and every  $w \in A_p$ ,*

$$T : L^p(w) \longrightarrow L^p(w)$$

*is bounded with constant controlled by*

$$(1.1) \quad C_1 \varphi \left( \frac{C_2 \|w\|_{A_p}^{\frac{p_0-1}{p-1}}}{(p-1)^{p_0-1}} \right), \quad \text{as } p \rightarrow 1^+.$$

Here  $C_1$  and  $C_2$  are two positive constants independent of  $p$  and  $w$ .

At this point we should emphasize that, under the hypotheses of Theorem 1.2, it is not possible to extrapolate down to  $p = 1$ , not even if we are only seeking a weak-type estimate without weights. To illustrate this limitation, we can consider the composition  $M^2 = M \circ M$ . Clearly,  $M^2 : L^p(w) \rightarrow L^p(w)$  for every  $1 < p < \infty$  and  $w \in A_p$ , but  $M^2 : L^1(\mathbb{R}^n) \not\rightarrow L^{1,\infty}(\mathbb{R}^n)$ . Therefore, finding endpoint estimates (close to  $L^1$ ) for operators under the assumptions of Rubio de Francia's Theorem 1.2 becomes an interesting goal.

Looking at Theorem 1.2, one can immediately see that, if  $T$  satisfies its hypotheses with  $\varphi(t) = t^s$  for some  $s > 0$ , then

$$(1.2) \quad T : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad \|T\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \leq \frac{C}{(p-1)^{s(p_0-1)}}.$$

The problem of approaching the endpoint  $L^1$  only from information in  $L^p$  for  $p > 1$  such as the one obtained in (1.2) is the starting point of another extrapolation theory, initiated by S. Yano in 1951. His original result [40] can be stated as follows:

**Theorem 1.3.** *Fix  $(\Omega, \mu)$  a finite measure space,  $p_0 > 1$  and  $m > 0$ . If  $T$  is a sublinear operator such that, for every  $1 < p \leq p_0$ ,*

$$T : L^p(\mu) \longrightarrow L^p(\mu)$$

*is bounded with norm controlled by  $C(p-1)^{-m}$ , for some  $C > 0$  independent of  $p$ , then,*

$$T : L(\log L)^m(\mu) \longrightarrow L^1(\mu)$$

*is also bounded.*

Recall that  $L(\log L)^m(\mu) \subsetneq L^1(\mu)$  is the space of  $\mu$ -measurable functions such that

$$\|f\|_{L(\log L)^m(\mu)} = \int_0^\infty f_\mu^*(t) \left(1 + \log_+ \frac{1}{t}\right)^m dt < \infty,$$

where, as usual,  $\log_+$  denotes the positive part of the logarithm and  $f_\mu^*$  is the decreasing rearrangement of  $f$  with respect to  $\mu$  defined by

$$f_\mu^*(t) = \inf\{y > 0 : \lambda_f^\mu(y) \leq t\}, \quad \lambda_f^\mu(t) := \mu(\{x \in \Omega : |f(x)| > t\}).$$

Theorem 1.3 can be extended to  $\sigma$ -finite measures and, also, improved in order to have weaker hypotheses and a better range space. See [7, 8] for more details on this extension. Before we make its statement precise, let us recall that, given  $1 \leq p < \infty$  and  $0 < q < \infty$ , the Lorentz spaces  $L^{p,q}(\mu)$  are defined as the set of  $\mu$ -measurable functions such that

$$\|f\|_{L^{p,q}(\mu)} = \left(p \int_0^\infty (t \lambda_f^\mu(t)^{1/p})^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^\infty (t^{1/p} f_\mu^*(t))^q \frac{dt}{t}\right)^{1/q} < \infty,$$

and

$$\|f\|_{L^{p,\infty}(\mu)} = \sup_{t>0} t \lambda_f^\mu(t)^{1/p} = \sup_{t>0} t^{1/p} f_\mu^*(t).$$

Notice that if  $p = q$ , then  $L^{p,p}(\mu) = L^p(\mu)$ , and for every  $1 \leq p < \infty$ , we have the following chain of continuous inclusions:

$$L^{p,q_1}(\mu) \subseteq L^{p,q_2}(\mu) \subseteq L^{p,\infty}(\mu), \quad 0 < q_1 < q_2 < \infty.$$

We will also need to consider general log-type spaces. For simplicity, we will adopt the following notation:

$$\log_1(x) = 1 + \log_+(x) \quad \text{and} \quad \log_k(x) = \log_1 \log_{k-1}(x), \quad \text{for } k > 1.$$

For natural numbers  $1 \leq j_1 < j_2 < \dots < j_n$  and positive real numbers  $m_1, \dots, m_n > 0$ , we define the space

$$L(\log_{j_1} L)^{m_1} \dots (\log_{j_n} L)^{m_n}(\mu)$$

as the set of  $\mu$ -measurable functions such that

$$\|f\|_{L(\log_{j_1} L)^{m_1} \dots (\log_{j_n} L)^{m_n}(\mu)} = \int_0^\infty f_\mu^*(t) \log_{j_1}^{m_1} \frac{1}{t} \dots \log_{j_n}^{m_n} \frac{1}{t} dt < \infty.$$

This is the modern version of Theorem 1.3 that we will use:

**Theorem 1.4.** *Given a  $\sigma$ -finite measure  $\mu$  and  $m > 0$ , if a sublinear operator  $T$  satisfies that*

$$T : L^{p,1}(\mu) \longrightarrow L^p(\mu)$$

*is bounded with constant less than or equal to  $C(p-1)^{-m}$  for every  $1 < p \leq p_0$ , then*

$$T : L(\log L)^m(\mu) \longrightarrow E_m(\mu)$$

*is also bounded with constant essentially controlled by  $C$ , where  $E_m(\mu)$  is the space of  $\mu$ -measurable functions such that*

$$\|f\|_{E_m(\mu)} = \sup_{t>0} \frac{t f_\mu^{**}(t)}{\log_1^m t} < \infty,$$

*and  $f_\mu^{**}(t) = \frac{1}{t} \int_0^t f_\mu^*(s) ds$ .*

Therefore, as a consequence of (1.2) and Theorem 1.4, we can immediately deduce the following endpoint result for Rubio de Francia operators:

**Corollary 1.5.** *Let  $1 < p_0 < \infty$ ,  $s > 0$  and let  $T$  be a sublinear operator such that*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

*is bounded for every  $w \in A_{p_0}$  with constant  $C\|w\|_{A_{p_0}}^s$ . Then,*

$$T : L(\log L)^{s(p_0-1)}(\mathbb{R}^n) \longrightarrow E_{s(p_0-1)}(\mathbb{R}^n)$$

*is also bounded.*

Estimates of the type (1.2) (or similar ones when  $p \rightarrow \infty$ ) for operators satisfying the hypotheses of Theorem 1.2 with  $\varphi(t) = t^s$  have appeared in different situations in the literature (see, for example, [19, 22, 32]). In particular, in [32], a converse argument can be used to find optimal values of  $s$ .

However, as we shall see in this paper, there is still much more to say. Our main goal is to obtain endpoint results close to  $L^1(u)$  for Rubio de Francia operators with respect to  $A_1$  weights. To this end, we will combine several variants of both Rubio de Francia and Yano extrapolation theories, and then study under which Rubio de Francia condition on  $T$  we can obtain the best endpoint estimate. The paper is organized as follows: in Section 2 we present our main results for operators under similar assumptions to those in Rubio de Francia's theorem. Then we will see how different types of operator norms associated with them blow up as  $p$  tends to 1, and how this information can be exploited to extrapolate in the sense of Yano. Next, in Section 3 we improve a particular endpoint estimate which is related to the extrapolation theory introduced in [10]. Finally, in Section 4, we see how our results can be useful in different applications.

From now on, we will write  $x \lesssim y$  when there is a positive constant  $C > 0$  such that  $x \leq Cy$ . If both  $x \lesssim y$  and  $y \lesssim x$ , then we write  $x \approx y$ . The constants involved do not depend on any parameter that is not fixed in its context.

## 2. ENDPOINT ESTIMATES FOR RUBIO DE FRANCIA OPERATORS

Given a sublinear operator  $T$  defined on the set of  $\mu$ -measurable functions, we say that  $T$  is of strong-type  $(p_0, p_0)$  if

$$T : L^{p_0}(\mu) \longrightarrow L^{p_0}(\mu).$$

It is of weak-type  $(p_0, p_0)$  if

$$T : L^{p_0}(\mu) \longrightarrow L^{p_0, \infty}(\mu).$$

It is of restricted weak-type  $(p_0, p_0)$  if

$$T : L^{p_0, 1}(\mu) \longrightarrow L^{p_0, \infty}(\mu),$$

and finally, of strong weak-type  $(p_0, p_0)$  if

$$T : L^{p_0, \infty}(\mu) \longrightarrow L^{p_0, \infty}(\mu).$$

In the top part of Table 1, we summarize the four basic cases that we will study in this section. Notice that the assumptions (first row) will be some of the previous boundedness at level  $p_0 > 1$  with respect to  $A_{p_0}$  weights, and with constant essentially controlled by a power of  $\|w\|_{A_{p_0}}$ . From here, we will deduce the same boundedness but for every  $1 < p < p_0$  with respect to  $A_1$  weights, keeping track of how the constant blows up when  $p \rightarrow 1$  (second row). This information will then be used to extrapolate in the sense of Yano with a suitable technique, and reach an endpoint space of logarithmic type with  $A_1$  weights (third row). In the bottom part of the table, we also show what we obtain if we go from the hypotheses of Subsection 2.3 to those in Subsection 2.4, and then extrapolate from there.

| Subsection 2.1   | Subsection 2.2  | Subsection 2.3  | Subsection 2.4  |
|--|---|---|---|
| $T : L^{p_0}(w) \rightarrow L^{p_0}(w)$<br>$\ T\  \lesssim \ w\ _{A_{p_0}}^s$  | $T : L^{p_0}(w) \rightarrow L^{p_0, \infty}(w)$<br>$\ T\  \lesssim \ w\ _{A_{p_0}}^\sigma$    | $T : L^{p_0, 1}(w) \rightarrow L^{p_0, \infty}(w)$<br>$\ T\  \lesssim \ w\ _{A_{p_0}}^r$                        | $T : L^{p_0, \infty}(w) \rightarrow L^{p_0, \infty}(w)$<br>$\ T\  \lesssim \ w\ _{A_{p_0}}^\alpha$      |
| $T : L^p(u) \rightarrow L^p(u)$<br>$\ T\  \lesssim \frac{1}{(p-1)^{s(p_0-1)}}$ | $T : L^p(u) \rightarrow L^{p, \infty}(u)$<br>$\ T\  \lesssim \frac{1}{(p-1)^{\sigma(p_0-1)}}$ | $T : L^{p, p/p_0}(u) \rightarrow L^{p, \infty}(u)$<br>$\ T\  \lesssim \frac{1}{(p-1)^{r(p_0-1)}}$               | $T : L^{p, \infty}(u) \rightarrow L^{p, \infty}(u)$<br>$\ T\  \lesssim \frac{1}{(p-1)^{\alpha(p_0-1)}}$ |
| $L(\log L)^{s(p_0-1)}(u)$  | $L(\log L)^{\sigma(p_0-1)} \log_3 L(u)$   | $L(\log L)^{r(p_0-1)+b} \log_3 L(u)$<br>$b = \min(1, p_0 - 1)$<br>$(\varepsilon, \delta)$ -atomic case: $b = 0$ | $[L(\log L)^{\alpha(p_0-1)-1} \log_3 L(u)]_1$   |

| Subsection 2.5   |
|--|
| $T : L^{p_0, 1}(w) \rightarrow L^{p_0, \infty}(w)$<br>$\ T\  \lesssim \ w\ _{A_{p_0}}^r$               |
| $T : L^{p_0, \infty}(w) \rightarrow L^{p_0, \infty}(w)$<br>$\ T\  \lesssim \ w\ _{A_{p_0}}^{r+p_0'-1}$ |
| $[L(\log L)^{r(p_0-1)} \log_3 L(u)]_1$   |

TABLE 1. In all the cases,  $w \in A_{p_0}$ ,  $u \in A_1$  and  $1 < p < p_0$ .

For later purposes, let us mention that, using that

$$M : L^1(u) \rightarrow L^{1,\infty}(u), \quad \|M\|_{L^1(u) \rightarrow L^{1,\infty}(u)} \lesssim \|u\|_{A_1},$$

and that  $M$  is bounded on  $L^\infty$  with constant 1, one can easily prove that, for every  $u \in A_1$ ,

$$(Mf)_u^*(t) \lesssim \frac{\|u\|_{A_1}}{t} \int_0^t f_u^*(s) ds.$$

From here, it follows that

$$(2.1) \quad \|M\|_{L^p(u) \rightarrow L^p(u)} \lesssim \frac{\|u\|_{A_1}}{p-1}, \quad \|M\|_{L^{p,\infty}(u) \rightarrow L^{p,\infty}(u)} \lesssim \frac{\|u\|_{A_1}}{p-1}.$$

Also, the sharp weak-type  $(p, p)$  estimate for  $M$  due to S. M. Buckley [6] states that

$$(2.2) \quad \|M\|_{L^p(w) \rightarrow L^{p,\infty}(w)} \lesssim \|w\|_{A_p}^{1/p}.$$

**2.1. From strong-type to strong-type.** In this subsection, our starting hypothesis will be the classical one in Rubio de Francia's extrapolation theory. That is, for some  $1 < p_0 < \infty$  and every  $w \in A_{p_0}$ ,  $T$  is of strong type  $(p_0, p_0)$  with respect to the weight  $w$ , with constant  $\varphi(\|w\|_{A_{p_0}})$ .

From (1.1), we can see that, even if  $\varphi(t) = t^s$  for some  $s > 0$ , the blow-up of the constant is exponential except in the unweighted case, and thus Yano's theory cannot be applied directly. To avoid this problem, our first result is an easy modification of the proof of Theorem 1.2 in [21]. We believe it is well-known to the experts in the topic but we have not found a proof in the literature and hence we include it for the sake of completeness.

**Theorem 2.1.** *Given an operator  $T$ , if for some  $1 < p_0 < \infty$  and every  $w \in A_{p_0}$ ,*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

*is bounded with constant  $\varphi(\|w\|_{A_{p_0}})$ , then, given  $u \in A_1$ ,*

$$T : L^p(u) \longrightarrow L^p(u)$$

*is bounded for every  $1 < p < p_0$  with constant controlled by*

$$(2.3) \quad C_1 \min \left[ \varphi \left( \frac{C_2 \|u\|_{A_1}^{p_0}}{(p-1)^{p_0-1}} \right), \left( \frac{\|u\|_{A_1}}{p-1} \right)^{1-\frac{p}{p_0}} \varphi \left( \frac{C_3 \|u\|_{A_1}}{(p-1)^{p_0-1}} \right) \right],$$

*for some universal constants  $C_1, C_2$  and  $C_3$ .*

*Proof.* In order to get the behavior of the constant, we shall proceed in two different ways:

1) Given  $u \in A_1$ , let us consider the Rubio de Francia algorithm:

$$Rf(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{2^k \|M^k\|_{L^p(u) \rightarrow L^p(u)}}.$$

Then, clearly,  $f \leq Rf$ ,  $Rf \in A_1$  and using (2.1) we obtain that

$$\|Rf\|_{A_1} \leq 2 \|M\|_{L^p(u) \rightarrow L^p(u)} \lesssim \frac{\|u\|_{A_1}}{p-1},$$

and

$$\|Rf\|_{L^p(u)} \leq 2\|f\|_{L^p(u)}.$$

Therefore,

$$\begin{aligned} \int |Tf(x)|^p u(x) dx &= \int |Tf(x)|^p (Rf(x))^{\frac{p}{p_0}(p-p_0)} (Rf(x))^{\frac{p}{p_0}(p_0-p)} u(x) dx \\ &\leq \left( \int |Tf(x)|^{p_0} (Rf(x))^{p-p_0} u(x) dx \right)^{\frac{p}{p_0}} \left( \int (Rf(x))^p u(x) dx \right)^{1-\frac{p}{p_0}} \\ &\leq 2^{\frac{p(p_0-p)}{p_0}} \varphi(\|(Rf)^{p-p_0} u\|_{A_{p_0}})^p \left( \int |f(x)|^{p_0} (Rf(x))^{p-p_0} u(x) dx \right)^{\frac{p}{p_0}} \left( \int |f(x)|^p u(x) dx \right)^{1-\frac{p}{p_0}} \\ &\lesssim \varphi(\|(Rf)^{p-p_0} u\|_{A_{p_0}})^p \left( \int |f(x)|^p u(x) dx \right), \end{aligned}$$

and we obtain that  $T : L^p(u) \rightarrow L^p(u)$  with constant less than or equal to the first term in (2.3), since

$$\|(Rf)^{p-p_0} u\|_{A_{p_0}} \lesssim \|Rf\|_{A_1}^{p_0-1} \|u\|_{A_1} \lesssim \frac{\|u\|_{A_1}^{p_0}}{(p-1)^{p_0-1}}.$$

2) To get the second term in (2.3), we avoid the use of Rubio de Francia's algorithm:

$$\begin{aligned} \int |Tf(x)|^p u(x) dx &= \int |Tf(x)|^p (Mf(x))^{\frac{p}{p_0}(p-p_0)} (Mf(x))^{\frac{p}{p_0}(p_0-p)} u(x) dx \\ &\leq \left( \int |Tf(x)|^{p_0} (Mf(x))^{p-p_0} u(x) dx \right)^{\frac{p}{p_0}} \left( \int (Mf(x))^p u(x) dx \right)^{1-\frac{p}{p_0}}. \end{aligned}$$

Now,  $w := (Mf)^{p-p_0} u \in A_{p_0}$  with

$$\begin{aligned} (2.4) \quad \|w\|_{A_{p_0}} &= \left\| \left[ (Mf)^{\frac{p_0-p}{p_0-1}} \right]^{1-p_0} u \right\|_{A_{p_0}} \leq \left\| (Mf)^{\frac{p_0-p}{p_0-1}} \right\|_{A_1}^{p_0-1} \|u\|_{A_1} \\ &\lesssim \left( \frac{p_0-1}{p-1} \right)^{p_0-1} \|u\|_{A_1}. \end{aligned}$$

Hence, we can use our assumption and (2.1) to deduce that

$$\begin{aligned} &\int |Tf(x)|^p u(x) dx \\ &\lesssim \varphi(\|w\|_{A_{p_0}})^p \left( \int |f(x)|^{p_0} (Mf(x))^{p-p_0} u(x) dx \right)^{\frac{p}{p_0}} \left( \left( \frac{\|u\|_{A_1}}{p-1} \right)^p \int |f(x)|^p u(x) dx \right)^{1-\frac{p}{p_0}} \\ &\leq \left( \frac{\|u\|_{A_1}}{p-1} \right)^{p(1-\frac{p}{p_0})} \varphi\left( \frac{C\|u\|_{A_1}}{(p-1)^{p_0-1}} \right)^p \left( \int |f(x)|^p u(x) dx \right), \end{aligned}$$

and the result follows taking the minimum of both constants.  $\square$

We observe that, since we are interested in getting the slowest blow-up possible in terms of  $p$ , the constant obtained by the first procedure is better for our purposes. However, in standard cases ( $p_0 = 2$ ,  $\varphi(t) = t$ ), it gives a worst exponent in  $\|u\|_{A_1}$ .

As a first consequence, we obtain the following extension of Corollary 1.5 for weights  $u \in A_1$ .

**Corollary 2.2.** *Let  $1 < p_0 < \infty$ ,  $s > 0$  and let  $T$  be a sublinear operator such that*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

*is bounded for every  $w \in A_{p_0}$  with constant  $C\|w\|_{A_{p_0}}^s$ . Then, for every  $u \in A_1$ ,*

$$T : L(\log L)^{s(p_0-1)}(u) \longrightarrow E_{s(p_0-1)}(u)$$

*is also bounded with constant less than or equal to  $C\|u\|_{A_1}^{sp_0}$ .*

**Remark 2.3.** *We shall keep the letter  $s$  for the exponent of  $\|w\|_{p_0}$  in the strong-type  $(p_0, p_0)$  case.*

## 2.2. From weak-type to weak-type.

**Theorem 2.4.** *Let  $T$  be an operator such that, for some  $1 < p_0 < \infty$  and every  $w \in A_{p_0}$ ,*

$$T : L^{p_0}(w) \longrightarrow L^{p_0, \infty}(w)$$

*is bounded with constant  $\varphi(\|w\|_{A_{p_0}})$ . Then, for every  $1 < p < p_0$  and every  $u \in A_1$ ,*

$$T : L^p(u) \longrightarrow L^{p, \infty}(u)$$

*is bounded with constant controlled by*

$$C_1 \|u\|_{A_1}^{\frac{1}{p} - \frac{1}{p_0}} \varphi \left( C_2 \left( \frac{p_0 - 1}{p - 1} \right)^{p_0 - 1} \|u\|_{A_1} \right).$$

*Proof.* Let  $\gamma > 0$  and  $y > 0$ ,

$$\lambda_{Tf}^u(y) \leq \lambda_{Mf}^u(\gamma y) + \gamma^{p_0-p} \frac{y^{p_0}}{y^p} \int_{\{|Tf|>y\}} (Mf)^{p-p_0}(x) u(x) dx.$$

Then, by hypothesis, if  $w = (Mf)^{p-p_0}u$  we deduce that

$$\begin{aligned} \lambda_{Tf}^u(y) &\leq \lambda_{Mf}^u(\gamma y) + \gamma^{p_0-p} \frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p} \int |f(x)|^{p_0} (Mf)^{p-p_0} u(x) dx \\ &\leq \lambda_{Mf}^u(\gamma y) + \gamma^{p_0-p} \frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p} \int |f(x)|^p u(x) dx. \end{aligned}$$

Using (2.2), we have that

$$\lambda_{Mf}^u(\gamma y) \lesssim \frac{\|u\|_{A_1}}{y^p \gamma^p} \|f\|_{L^p(u)}^p.$$

Combining these two facts and multiplying by  $y^p$  we obtain

$$y^p \lambda_{Tf}^u(y) \lesssim \left( \frac{\|u\|_{A_1}}{\gamma^p} + \gamma^{p_0-p} \varphi(\|w\|_{A_{p_0}})^{p_0} \right) \|f\|_{L^p(u)}^p.$$

Finally, we can minimize the right-hand side with respect to  $\gamma > 0$  by choosing  $\gamma = \|u\|_{A_1}^{1/p_0} \varphi(\|w\|_{A_{p_0}})^{-1}$ , and taking supremum over  $y > 0$ , we get,

$$\|Tf\|_{L^{p, \infty}(u)}^p \lesssim \|u\|_{A_1}^{1-p/p_0} \varphi(\|w\|_{A_{p_0}})^p \|f\|_{L^p(u)}^p.$$

This estimate, together with (2.4), completes the proof.  $\square$



**Corollary 2.5.** *Under the hypotheses of Theorem 2.4, with  $\varphi(t) = t^\sigma$  for some  $\sigma > 0$ , we obtain that, for every  $u \in A_1$ ,*

$$T : L^p(u) \longrightarrow L^{p,\infty}(u)$$

*is bounded with constant less than or equal to*

$$C \frac{\|u\|_{A_1}^{\sigma + \frac{1}{p} - \frac{1}{p_0}}}{(p-1)^{\sigma(p_0-1)}}.$$

At this point, we have to use the first variant of Yano's extrapolation theorem concerning weak-type spaces. In 1996, N. Yu Antonov [1] proved that there is almost everywhere convergence for the Fourier series of every function in  $L \log L \log_3 L(\mathbb{T})$ . Even though he did not write it explicitly, behind his ideas there was the following extrapolation argument (see [2, 11, 12, 37] for more details):

**Theorem 2.6.** *Let  $1 < p_0 < \infty$  and  $m > 0$ . If  $T$  is a sublinear operator such that*

$$(2.5) \quad T : L^p(\mu) \longrightarrow L^{p,\infty}(\mu)$$

*is bounded with constant controlled by  $C(p-1)^{-m}$  for every  $1 < p \leq p_0$ , then*

$$T : L(\log L)^m \log_3 L(\mu) \longrightarrow R_m(\mu)$$

*is also bounded with constant essentially controlled by  $C$ , where  $R_m(\mu)$  is the space of  $\mu$ -measurable functions such that*

$$\|f\|_{R_m(\mu)} = \sup_{t>0} \frac{t f_\mu^*(t)}{\log_1^m t} < \infty.$$

**Corollary 2.7.** *Under the hypotheses of Theorem 2.4 with  $\varphi(t) = t^\sigma$ , we obtain that, for every  $u \in A_1$ ,*

$$T : L(\log L)^{\sigma(p_0-1)} \log_3 L(u) \longrightarrow R_{\sigma(p_0-1)}(u)$$

*is bounded with constant less than or equal to  $C\|u\|_{A_1}^{\sigma+1-\frac{1}{p_0}}$ .*

**Remark 2.8.** *We shall keep the letter  $\sigma$  for the exponent of  $\|w\|_{p_0}$  in the weak-type  $(p_0, p_0)$  case. Clearly, for a given operator,  $\sigma \leq s$ .*

**Remark 2.9.** *It is important to mention that, in order to prove Theorem 2.6, we only need to assume that the sublinear operator  $T$  satisfies, for every function  $f$  such that  $\|f\|_\infty \leq 1$ ,*

$$(2.6) \quad \|Tf\|_{L^{p,\infty}(\mu)} \lesssim \frac{1}{(p-1)^m} \|f\|_{L^1(\mu)}^{1/p}.$$

*From here, the boundedness on  $L(\log L)^m \log_3 L(\mu)$  is obtained by expressing  $f$  as an appropriate linear combination of functions bounded by 1. This estimate (2.6) can be deduced from (2.5), but it also follows from the weaker boundedness*

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\mu),$$

*with constant  $C(p-1)^{-m}$ .*

### 2.3. From restricted weak-type to restricted weak-type.

**Theorem 2.10.** *Let  $T$  be an operator such that, for some  $1 < p_0 < \infty$  and every  $w \in A_{p_0}$ ,*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

*is bounded with constant  $\varphi(\|w\|_{A_{p_0}})$ . Then, for every  $1 < p < p_0$  and every  $u \in A_1$ ,*

$$T : L^{p,\frac{p}{p_0}}(u) \longrightarrow L^{p,\infty}(u)$$

*is bounded with constant controlled by*

$$C_1 \|u\|_{A_1}^{\frac{1}{p} - \frac{1}{p_0}} \varphi \left( C_2 \left( \frac{p_0 - 1}{p - 1} \right)^{p_0 - 1} \|u\|_{A_1} \right).$$

The proof follows the same pattern as in Theorem 2.4 with the obvious modifications. However, since the result is not the one we can initially expect and we have to be careful with the behavior of the constant, we include the details.

*Proof.* Let  $\gamma > 0$  and  $y > 0$ . Then, if  $w = (Mf)^{p-p_0}u$ ,

$$\begin{aligned} \lambda_{Tf}^u(y) &\leq \lambda_{Mf}^u(\gamma y) + \gamma^{p_0-p} \frac{y^{p_0}}{y^p} \int_{\{|Tf|>y\}} w(x) dx \\ &\lesssim \lambda_{Mf}^u(\gamma y) + \gamma^{p_0-p} \frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p} \left( \int_0^\infty \left( \int_{\{|f|>z\}} w(x) dx \right)^{1/p_0} dz \right)^{p_0}. \end{aligned}$$

But, since  $p - p_0 < 0$ , we can bound  $w = (Mf)^{p-p_0}u \leq z^{p-p_0}u$  on the set  $\{|f| > z\}$ , so we conclude that

$$\begin{aligned} \lambda_{Tf}^u(y) &\lesssim \lambda_{Mf}^u(\gamma y) + \gamma^{p_0-p} \frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p} \left( \int_0^\infty z^{\frac{p}{p_0}-1} \left( \int_{\{|f|>z\}} u(x) dx \right)^{1/p_0} dz \right)^{p_0} \\ &\approx \lambda_{Mf}^u(\gamma y) + \gamma^{p_0-p} \frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p} \|f\|_{L^{p,\frac{p}{p_0}}(u)}^p. \end{aligned}$$

The result follows as in Theorem 2.4, using (2.2) and the fact that  $\|f\|_{L^p(u)} \leq \|f\|_{L^{p,\frac{p}{p_0}}(u)}$ .  $\square$

**Lemma 2.11.** *If  $T$  is a sublinear operator such that, for every  $1 < p < p_0$ ,*

$$T : L^{p,\frac{p}{p_0}}(u) \longrightarrow L^{p,\infty}(u)$$

*is bounded with constant less than or equal to  $C_u(p-1)^{-m}$ , then:*

(i) *It holds that*

$$T : L^{p,1}(u) \longrightarrow L^{p,\infty}(u)$$

*is bounded with constant less than or equal to  $CC_u(p-1)^{-m-1}$ .*

(ii) *For every function  $f$  such that  $\|f\|_\infty \leq 1$ ,*

$$\|Tf\|_{L^{p,\infty}(u)} \lesssim \frac{C_u}{(p-1)^{m+\frac{p_0-1}{p}}} \|f\|_{L^1(u)}^{1/p}.$$

*Proof.* (i) The first result is well-known since, for every measurable set  $E$ ,

$$(2.7) \quad \|T\chi_E\|_{L^{p,\infty}(u)} \lesssim \frac{1}{(p-1)^m} u(E),$$

and  $L^{p,\infty}(u)$  can be endowed with a norm  $\|\cdot\|_*$  such that

$$\|f\|_{L^{p,\infty}} \leq \|f\|_* \leq \frac{1}{p-1} \|f\|_{L^{p,\infty}},$$

and hence the above estimate over measurable sets can be extended to any function by adding a factor  $\frac{1}{p-1}$  to the constant.

(ii) Let  $f$  be bounded by 1. Then

$$\begin{aligned} \|f\|_{L^{p,\frac{p}{p_0}}(u)} &= \left( p \int_0^1 \lambda_f^u(t)^{1/p_0} t^{\frac{p}{p_0}-1} dt \right)^{p_0/p} \lesssim \left( \int_0^1 \lambda_f^u(t) dt \right)^{1/p} \left( \int_0^1 t^{\frac{p-p_0}{p_0-1}} dt \right)^{\frac{p_0-1}{p}} \\ &\lesssim \|f\|_{L^1(u)}^{1/p} \left( \frac{p_0-1}{p-1} \right)^{\frac{p_0-1}{p}}, \end{aligned}$$

and the result follows.  $\square$

As a consequence of Theorem 2.10, Lemma 2.11 and Remark 2.9, we obtain the following endpoint estimate:

**Corollary 2.12.** *Let  $1 < p_0 < \infty$ ,  $r > 0$  and let  $T$  be a sublinear operator such that*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

*is bounded for every  $w \in A_{p_0}$  with constant  $C\|w\|_{A_{p_0}}^r$ . Then, for every  $u \in A_1$ ,*

$$T : L(\log L)^\beta \log_3 L(u) \longrightarrow R_\beta(u)$$

*is also bounded with constant  $C\|u\|_{A_1}^{r+1-\frac{1}{p_0}}$ , where  $\beta = r(p_0-1) + \min(1, p_0-1)$ .*

**Remark 2.13.** *We shall keep the letter  $r$  for the exponent of  $\|w\|_{p_0}$  in the restricted weak-type  $(p_0, p_0)$  case. Clearly, for a given operator,  $r \leq \sigma \leq s$ .*

**Remark 2.14.** *There is large class of operators (called  $(\varepsilon, \delta)$ -atomic approximable, see Definition 3.2 and [9, 10]) for which an estimate of the form (2.7) implies that, for every function  $f$  bounded by 1,*

$$\|Tf\|_{L^{p,\infty}(u)} \lesssim \frac{1}{(p-1)^m} \|f\|_{L^1(u)}^{1/p}.$$

*Hence, by Remark 2.9, we obtain that if an operator  $T$  in this class satisfies the hypotheses of Corollary 2.12, then*

$$T : L(\log L)^{r(p_0-1)} \log_3 L(u) \longrightarrow R_{r(p_0-1)}(u).$$

#### 2.4. From strong weak-type to strong weak-type.

**Theorem 2.15.** *Let  $T$  be an operator such that, for some  $1 < p_0 < \infty$  and every  $w \in A_{p_0}$ ,*

$$T : L^{p_0, \infty}(w) \longrightarrow L^{p_0, \infty}(w)$$

*is bounded with constant  $\varphi(\|w\|_{A_{p_0}})$ . Then, for every  $1 < p < p_0$  and every  $u \in A_1$ ,*

$$T : L^{p, \infty}(u) \longrightarrow L^{p, \infty}(u)$$

*is bounded with constant*

$$C_1 \varphi \left( \frac{C_2 \|u\|_{A_1}^{p_0}}{(p-1)^{p_0-1}} \right).$$

*Proof.* We use Rubio de Francia's algorithm as follows:

$$Rf(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{2^k \|M^k\|_{L^{p, \infty}(u) \rightarrow L^{p, \infty}(u)}}.$$

Then, clearly,  $f \leq Rf$ ,  $Rf \in A_1$  and using (2.1) we obtain that

$$\|Rf\|_{A_1} \leq 2 \|M\|_{L^{p, \infty}(u) \rightarrow L^{p, \infty}(u)} \lesssim \frac{\|u\|_{A_1}}{p-1}$$

and

$$\|Rf\|_{L^{p, \infty}(u)} \leq 2 \|f\|_{L^{p, \infty}(u)}.$$

Let  $\gamma > 0$  and  $y > 0$ ,

$$\lambda_{Tf}^u(y) \leq \lambda_{Rf}^u(\gamma y) + \gamma^{p_0-p} \frac{y^{p_0}}{y^p} \int_{\{|Tf|>y\}} (Rf)^{p-p_0}(x) u(x) dx.$$

Then, by hypothesis, we deduce that if  $w = (Rf)^{p-p_0} u$ ,

$$\begin{aligned} \lambda_{Tf}^u(y) &\leq \lambda_{Rf}^u(\gamma y) + \gamma^{p_0-p} \frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p} \sup_{z>0} z^{p_0} \int_{\{|f|>z\}} Rf(x)^{p-p_0} u(x) dx \\ &\leq \lambda_{Rf}^u(\gamma y) + \gamma^{p_0-p} \frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p} \sup_{z>0} z^p \int_{\{|f|>z\}} u(x) dx. \end{aligned}$$

Using that

$$\lambda_{Rf}^u(\gamma y) \lesssim \frac{2^p}{y^p \gamma^p} \|f\|_{L^{p, \infty}(u)}^p,$$

we obtain

$$y^p \lambda_{Tf}^u(y) \lesssim \left( \frac{1}{\gamma^p} + \gamma^{p_0-p} \varphi(\|w\|_{A_{p_0}})^{p_0} \right) \|f\|_{L^{p, \infty}(u)}^p,$$

and the result follows minimizing with  $\gamma = \varphi(\|w\|_{A_{p_0}})^{-1}$  and using the estimate analogous to (2.4) with  $R$  instead of  $M$ .  $\square$

The second variant of Yano's extrapolation theorem was done in [13]:

**Theorem 2.16.** *Let  $\mu$  be a  $\sigma$ -finite, non-atomic measure,  $1 < p_0 < \infty$ ,  $m > 0$ , and let*

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\mu)$$

*be a bounded sublinear operator with constant controlled by  $C(p-1)^{-m}$  for every  $1 < p \leq p_0$ . Then,*

$$T : [L(\log L)^{m-1} \log_3 L(\mu)]_1 \longrightarrow R_m(\mu),$$

*where  $X = [L(\log L)^{m-1} \log_3 L(\mu)]_1$  is the set of measurable functions such that*

$$\|f\|_X = \|f\|_{L^{1,\infty}(\mu)} + \int_0^1 \frac{\sup_{t \leq y} t f_\mu^*(t)}{y} \left( \log_1 \frac{1}{y} \right)^{m-1} \log_3 \frac{1}{y} dy < \infty.$$

**Remark 2.17.** *Here, we have to emphasize that*

$$L(\log L)^m \log_3 L(\mu) \subsetneq [L(\log L)^{m-1} \log_3 L(\mu)]_1.$$

*Thus, except for the  $\log_3 L$  factor, we can say that in order to obtain the best estimate from a Yano's extrapolation type theorem, it would be convenient to compute the best strong weak-type  $(p, p)$  constant for  $T$  as  $p \rightarrow 1$ .*

**Corollary 2.18.** *Let  $1 < p_0 < \infty$ , and let  $T$  be a sublinear operator such that*

$$T : L^{p_0,\infty}(w) \longrightarrow L^{p_0,\infty}(w)$$

*is bounded for every  $w \in A_{p_0}$  with constant  $C\|w\|_{A_{p_0}}^\alpha$ . Then, for every  $u \in A_1$ ,*

$$T : \left[ L(\log L)^{\alpha(p_0-1)-1} \log_3 L(u) \right]_1 \longrightarrow R_{\alpha(p_0-1)-1}(u)$$

*is also bounded.*

**Remark 2.19.** *We shall keep the letter  $\alpha$  for the exponent of  $\|w\|_{p_0}$  in the strong weak-type  $(p_0, p_0)$  case. Clearly, for a given operator,  $r \leq \sigma \leq \alpha$ , but there is no clear relation between  $\alpha$  and  $s$ .*

**2.5. From restricted weak-type to strong weak-type.** Taking into account Remarks 2.13 and 2.17, it is clear that a good way to obtain the best endpoint estimate would be to seek good restricted weak-type estimates for  $T$  with respect to every weight in  $A_{p_0}$  (that is, a small exponent  $r$ ) and from here, try to find a good constant for the strong weak-type  $(p, p)$  with respect to  $A_1$  weights. In other words, we want to find an optimal way to relate  $r$  and  $\delta$  in the following situation:

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w), \quad C\|w\|_{A_{p_0}}^r \implies T : L^{p,\infty}(u) \longrightarrow L^{p,\infty}(u), \quad \frac{C}{(p-1)^\delta}.$$

To this end, we shall use the following interpolation result which is proved in the Appendix.

**Lemma 2.20.** *Let  $0 < s_0, s_1 \leq 1 < q_0 < q_1 < \infty$  and let  $T$  be a sublinear operator such that, for some weight  $u$ ,*

$$T : L^{q_j, s_j}(u) \longrightarrow L^{q_j, \infty}(u)$$

*is bounded with constant  $M_j$ , for  $j = 0, 1$ . Then, for every  $0 < \theta < 1$ , if  $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ , we have that*

$$T : L^{q,\infty}(u) \longrightarrow L^{q,\infty}(u)$$

is bounded with constant controlled by  $BM_0^{1-\theta}M_1^\theta$ , where

$$B = \left( \frac{q_0 q}{s_0(q - q_0)} \right)^{1/s_0} + \left( \frac{q_1 q}{s_1(q_1 - q)} \right)^{1/s_1} + \left( \frac{q_1}{s_1} \right)^{1/s_1}$$

**Theorem 2.21.** *Let  $1 < p_0 < \infty$ , and let  $T$  be a sublinear operator such that*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

*is bounded for every  $w \in A_{p_0}$  with constant  $\varphi(\|w\|_{A_{p_0}})$ . Then, for every  $w \in A_{p_0}$*

$$T : L^{p_0,\infty}(w) \longrightarrow L^{p_0,\infty}(w)$$

*is bounded with constant less than or equal to  $C_1\|w\|_{A_{p_0}}^{p'_0-1}\varphi(C_2\|w\|_{A_{p_0}})$ .*

*Proof.* From our hypothesis, we can write that, for every measurable set  $E \subseteq \mathbb{R}^n$  and every  $w \in A_{p_0}$ ,

$$\|T\chi_E\|_{L^{p_0,\infty}(w)} \leq \varphi(\|w\|_{A_{p_0}})w(E)^{1/p_0}.$$

This estimate can be extrapolated up by means of the modern version of Rubio de Francia's theorem (see its statement in [21]) and, given  $\varepsilon > 0$ , we obtain that

$$\|T\chi_E\|_{L^{p_0+\varepsilon,\infty}(w)} \lesssim \varphi(C\|w\|_{A_{p_0+\varepsilon}})w(E)^{1/(p_0+\varepsilon)},$$

for  $w \in A_{p_0+\varepsilon}$ . In particular,

$$(2.8) \quad T : L^{p_0+\varepsilon,1}(w) \longrightarrow L^{p_0+\varepsilon,\infty}(w),$$

for every  $w \in A_{p_0}$  and constant controlled by

$$\frac{\varphi(C\|w\|_{A_{p_0}})}{p_0 + \varepsilon - 1} \lesssim \varphi(C\|w\|_{A_{p_0}}).$$

Now we want to extrapolate down to  $p_0 - \varepsilon$ . Fix  $w \in A_{p_0}$  and set  $\varepsilon = C(p_0 - 1)\|w\|_{A_{p_0}}^{1-p'_0}$  in such a way that  $w \in A_{p_0-2\varepsilon}$  and  $\|w\|_{A_{p_0-2\varepsilon}} \lesssim \|w\|_{A_{p_0}}$  (see [28]). Now we proceed as in Theorem 2.10. For every measurable set  $E \subseteq \mathbb{R}^n$ ,  $\gamma > 0$  and  $y > 0$ ,

$$\lambda_{T\chi_E}^w(y) \leq \lambda_{M\chi_E}^w(\gamma y) + \gamma^\varepsilon y^\varepsilon \int_{\{|T\chi_E|>y\}} (M\chi_E)^{-\varepsilon}(x)w(x)dx.$$

But, using [21, Lemma 2.1],

$$\|(M\chi_E)^{-\varepsilon}w\|_{A_{p_0}} \leq \|(M\chi_E)^{1/2}\|_{A_1}^{2\varepsilon}\|w\|_{A_{p_0-2\varepsilon}} \lesssim \|w\|_{A_{p_0}},$$

so we can use our hypothesis and (2.2) to deduce that

$$y^{p_0-\varepsilon}\lambda_{T\chi_E}^w(y) \lesssim \left( \frac{\|w\|_{A_{p_0}}}{\gamma^{p_0-\varepsilon}} + \gamma^\varepsilon\varphi(\|w\|_{A_{p_0}})^{p_0} \right) w(E).$$

Minimizing in  $\gamma > 0$  we conclude that

$$\|T\chi_E\|_{L^{p_0-\varepsilon,\infty}} \lesssim \|w\|_{A_{p_0}}^{\frac{\varepsilon}{p_0(p_0-\varepsilon)}}\varphi(\|w\|_{A_{p_0}})w(E)^{1/(p_0-\varepsilon)},$$

and hence

$$(2.9) \quad T : L^{p_0-\varepsilon,1}(w) \longrightarrow L^{p_0-\varepsilon,\infty}(w)$$

with constant

$$\frac{\|w\|_{A_{p_0}}^{\frac{\varepsilon}{p_0(p_0-\varepsilon)}} \varphi(\|w\|_{A_{p_0}})}{p_0 - \varepsilon - 1} \lesssim \varphi(\|w\|_{A_{p_0}}),$$

recalling that  $\varepsilon \approx (p_0 - 1)\|w\|_{A_{p_0}}^{1-p'_0}$ . Finally, we use Lemma 2.20 with (2.9) and (2.8) to conclude that

$$T : L^{p_0, \infty}(w) \longrightarrow L^{p_0, \infty}(w)$$

with constant controlled by  $\frac{C\varphi(C\|w\|_{A_{p_0}})}{\varepsilon} \approx C_{p_0}\|w\|_{A_{p_0}}^{p'_0-1}\varphi(C\|w\|_{A_{p_0}})$ .  $\square$

**Corollary 2.22.** *Let  $1 < p_0 < \infty$ , and let  $T$  be a sublinear operator such that*

$$T : L^{p_0, 1}(w) \longrightarrow L^{p_0, \infty}(w)$$

*is bounded for every  $w \in A_{p_0}$  with constant  $C\|w\|_{A_{p_0}}^r$ . Then, for every  $u \in A_1$ ,*

$$T : \left[ L(\log L)^{r(p_0-1)} \log_3 L(u) \right]_1 \longrightarrow R_{r(p_0-1)}(u)$$

*is also bounded.*

### 3. A DIFFERENT CLASS OF WEIGHTS

Very recently, the authors in [10] presented a different kind of extrapolation in the sense of Rubio de Francia that allows us to reach the endpoint  $p = 1$  at least when restricted to characteristic functions. More precisely, they introduced a new class of weights  $\widehat{A}_p$ , closely related to  $A_p$ , in such a way that, if an operator  $T$  satisfies

$$(3.1) \quad T : L^{p_0, 1}(w) \longrightarrow L^{p_0, \infty}(w),$$

for some  $1 < p_0 < \infty$  and every weight  $w \in \widehat{A}_{p_0}$ , then we can conclude that

$$\|T\chi_E\|_{L^{1, \infty}(u)} \lesssim u(E),$$

for every  $u \in A_1$  and every measurable set  $E \subseteq \mathbb{R}^n$ . The class  $\widehat{A}_p$  is obviously larger than  $A_p$ , because otherwise we would have, again,  $M^2$  as a counterexample. However, it holds that, for every  $1 \leq p < \infty$  and every  $\varepsilon > 0$ ,

$$A_p \subseteq \widehat{A}_p \subseteq A_{p+\varepsilon}.$$

The precise definition of these classes is

$$\widehat{A}_p = \{(Mf)^{1-p}u : f \in L^1_{\text{loc}}, u \in A_1\},$$

and they are a subclass of the so-called  $A_p^{\mathcal{R}}$  weights, introduced in 1982 by R. Kerman and A. Torchinsky [29] to characterize the boundedness (3.1) of the Hardy-Littlewood maximal operator  $M$ . The extrapolation presented in [10] is more general, but the most interesting part for our purposes can be stated as follows:

**Theorem 3.1.** *Let  $T$  be a sublinear operator such that, for some  $1 < p_0 < \infty$  and every  $w \in \widehat{A}_{p_0}$ , it holds that*

$$T : L^{p_0, 1}(w) \longrightarrow L^{p_0, \infty}(w).$$

*Then, for every  $u \in A_1$ ,*

$$(i) \quad \|T\chi_E\|_{L^{1, \infty}(u)} \lesssim u(E), \quad E \subseteq \mathbb{R}^n,$$

(ii)  $T : L(\log L)^\varepsilon(u) \longrightarrow L_{\text{loc}}^{1,\infty}(u)$ ,  $\varepsilon > 0$ .

It is known that, in general, the estimate (i) that they obtain at level  $p = 1$  cannot hold for every function  $f \in L^1(u)$ . Take, for instance, the operator

$$(3.2) \quad Af(x) = \left\| \frac{f(x+y)}{y} \right\|_{L^{1,\infty}(\mathbb{R})},$$

which was introduced in [3] and is related to Bourgain's return time theorems. It trivially satisfies  $A\chi_E \leq M\chi_E$  (and hence, the assumption of Theorem 3.1), but it is not of weak-type  $(1, 1)$ . However, it can be proved that, for a wide class of operators called  $(\varepsilon, \delta)$ -atomic approximable (see Definition 3.2 below), the estimate on characteristic functions (i) is in fact equivalent to the weighted weak-type  $(1, 1)$ .

**Definition 3.2.** *Given  $\delta > 0$ , a function  $a \in L^1(\mathbb{R}^n)$  is called a  $\delta$ -atom if it satisfies the following properties:*

- $\int_{\mathbb{R}^n} a = 0$ , and
- there exists a cube  $Q$  such that  $|Q| \leq \delta$  and  $\text{supp } a \subseteq Q$ .

With this, a sublinear operator  $T$  is  $(\varepsilon, \delta)$ -atomic if, for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\|Ta\|_{L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)} = \int_0^1 (Ta)^*(t)dt \leq \varepsilon \|a\|_{L^1(\mathbb{R}^n)},$$

for every  $\delta$ -atom  $a$ , and  $T$  is said to be  $(\varepsilon, \delta)$ -atomic approximable if there exists a sequence  $\{T_n\}_n$  of  $(\varepsilon, \delta)$ -atomic operators such that, for every measurable set  $E$ ,  $|T_n\chi_E| \leq |T\chi_E|$  and, for every function  $f \in L^1(\mathbb{R}^n)$  with  $\|f\|_\infty \leq 1$ ,

$$|Tf(x)| \leq \liminf_n |T_n f(x)|, \quad \text{a.e. } x \in \mathbb{R}^n.$$

In [9], the author shows that this is not a strong property to assume on an operator. For instance, it is checked that if

$$(3.3) \quad Tf(x) = K * f(x),$$

with  $K \in L^p(\mathbb{R}^n)$  for some  $1 \leq p < \infty$ , then  $T$  is  $(\varepsilon, \delta)$ -atomic, and if  $\{T_n\}_n$  is a sequence of  $(\varepsilon, \delta)$ -atomic operators, then  $\sup_n |T_n f(x)|$  is  $(\varepsilon, \delta)$ -atomic approximable. See [9, 10] for more examples.

Therefore, the conclusion (ii) of Theorem 3.1 is especially interesting for operators which are not  $(\varepsilon, \delta)$ -atomic approximable, since in this case, (ii) is the best endpoint result that is not restricted to characteristic functions. In this section, we will see that this can be improved to the larger space  $L \log_2 L(u)$ .

**Theorem 3.3.** *Let  $T$  be a sublinear operator such that, for some  $1 < p_0 < \infty$  and every  $w \in \widehat{A}_{p_0}$ , it holds that*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

*is bounded. Then, for every  $u \in A_1$ ,*

$$T : L \log_2 L(u) \longrightarrow L_{\text{loc}}^{1,\infty}(u)$$

*is also bounded.*

*Proof.* First, we apply Theorem 3.1, which yields



- (i)  $\|T\chi_E\|_{L^{1,\infty}(u)} \lesssim u(E)$ ,
- (ii)  $T : L(\log L)^\varepsilon(u) \longrightarrow L_{\text{loc}}^{1,\infty}(u)$ .

Since  $L^1(u) \cap L^\infty$  is continuously embedded in  $L(\log L)^\varepsilon(u)$ , from (ii) we deduce that,

$$(3.4) \quad T : L^1(u) \cap L^\infty \rightarrow L_{\text{loc}}^{1,\infty}(u).$$

Take now a non-negative function  $f = f_0 + f_1$ , with  $f_0 = f\chi_{\{f \leq 1\}}$  and  $f_1 = f\chi_{\{f > 1\}}$ . By sublinearity, we have that  $\|Tf\|_{L_{\text{loc}}^{1,\infty}(u)} \lesssim \|Tf_0\|_{L_{\text{loc}}^{1,\infty}(u)} + \|Tf_1\|_{L_{\text{loc}}^{1,\infty}(u)}$ . For the term with  $f_0$ , we use (3.4) and  $\|f_0\|_\infty \leq 1$  to get

$$\|Tf_0\|_{L_{\text{loc}}^{1,\infty}(u)} \lesssim \|f_0\|_{L^1(u)} + \|f_0\|_\infty \leq \|f\|_{L \log_2 L(u)} + 1.$$

Now, to deal with  $f_1$ , we need to resort to [38, Lemma 4], where the author presents the following decomposition for non-negative functions:

$$(3.5) \quad f_1(x) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} 2^k \chi_{E_{k,j}}(x) \quad \text{a.e. } x \in \mathbb{R}^n,$$

where the sets  $E_{k,j}$  depend on  $f_1$  and are defined in such a way that, for every weight (in particular  $u$ ),

$$u(E_{k,j}) \leq \lambda_{f_1}^u(2^{k+j}).$$

For every  $N > 0$ , set  $f_1^N$  to be the following truncated series:

$$f_1^N(x) = \sum_{j=1}^N \sum_{|k| \leq N} 2^k \chi_{E_{k,j}}(x).$$

Next, since it is a finite sum, we use the sublinearity of  $T$  and (i) in order to obtain

$$(3.6) \quad \|Tf_1^N\|_{L_{\text{loc}}^{1,\infty}(u)} \lesssim \sum_{j=1}^{\infty} \log_1(j) \sum_{k \in \mathbb{Z}} 2^k \log_1(|k|) \lambda_{f_1}^u(2^{k+j}).$$

The logarithmic terms come from Stein and Weiss' lemma for the  $L^{1,\infty}$  quasinorm (see [39]). Now, fix  $j \geq 1$  and split the inner sum into three pieces:  $I_j^1 + I_j^2 + I_j^3$ . The first one will be

$$I_j^1 = \sum_{k < -j} 2^k \log_1(|k|) \lambda_{f_1}^u(2^{k+j}) \leq \|f\|_{L \log_2 L(u)} \sum_{k=j+1}^{\infty} 2^{-k} \log_1(k).$$

Here we used that, since  $f_1 > 1$ , we have  $\lambda_{f_1}^u(2^{k+j}) \leq \|f\|_{L \log_2 L(u)}$  whenever  $k < -j$ , because in this case,

$$\lambda_{f_1}^u(2^{k+j}) = \lambda_{f_1}(1) \leq \|f\|_{L \log_2 L(u)}.$$

The second term we need to consider is

$$\begin{aligned} I_j^2 &= \sum_{k=-j}^0 2^k \log_1(|k|) \lambda_{f_1}^u(2^{k+j}) = 2^{-j} \sum_{k=-j}^0 2^{k+j} \log_1(|k|) \lambda_{f_1}^u(2^{k+j}) \\ &\leq 2^{-j} \|f\|_{L \log_2 L(u)} \sum_{k=0}^j \log_1(k). \end{aligned}$$

Here we just used that  $t\lambda_f^u(t) \leq \|f\|_{L\log_2 L(u)}$ , for every  $t > 0$ . Finally,

$$\begin{aligned} I_j^3 &= \sum_{k=1}^{\infty} 2^k \log_1(k) \lambda_{f_1}^u(2^{k+j}) \leq 2^{-j} \sum_{k=1}^{\infty} 2^{k+j} \log_1(k+j) \lambda_{f_1}^u(2^{k+j}) \\ &\lesssim 2^{-j} \int_0^{\infty} \lambda_f^u(s) \log_2(s) ds \lesssim 2^{-j} \|f\|_{L\log_2 L(u)}. \end{aligned}$$

Now we go back to (3.6) and using the bounds for  $I_j^m$ ,  $m = 1, 2, 3$ , we conclude that

$$\begin{aligned} \|Tf_1^N\|_{L_{\text{loc}}^{1,\infty}(u)} &\lesssim \|f\|_{L\log_2 L(u)} \sum_{j=1}^{\infty} \log_1(j) \left( \sum_{k=j+1}^{\infty} 2^{-k} \log_1(k) + 2^{-j} \sum_{k=0}^j \log_1(k) + 2^{-j} \right) \\ &\lesssim \|f\|_{L\log_2 L(u)}. \end{aligned}$$

Therefore, we have that  $\|Tf_1^N\|_{L_{\text{loc}}^{1,\infty}(u)} \lesssim \|f\|_{L\log_2 L(u)}$ . If we show that  $f_1^N$  converges to  $f_1$  in  $L\log_2 L(u)$ , then we conclude  $\|Tf_1\|_{L_{\text{loc}}^{1,\infty}(u)} \lesssim \|f\|_{L\log_2 L(u)}$  and hence,

$$\|Tf\|_{L_{\text{loc}}^{1,\infty}(u)} \lesssim \|f\|_{L\log_2 L(u)} + 1.$$

From here, we finish the proof changing  $f$  by  $\alpha f$  and letting  $\alpha$  tend to infinity. To show that  $f_1^N \rightarrow f_1$  in  $L\log_2 L(u)$ , we observe that the difference  $f_1(x) - f_1^N(x)$  decreases to zero for almost every  $x \in \mathbb{R}^n$ , since  $f_1^N$  is a partial sum of a convergent series of positive terms that coincides with  $f_1$  almost everywhere. In particular, its decreasing rearrangement with respect to  $u$  satisfies that

$$(f_1 - f_1^N)_u^*(t) \rightarrow 0, \quad \text{a.e. } t \in (0, \infty).$$

On the other hand,  $|f_1 - f_1^N|$  can be pointwise controlled by  $f_1 \in L\log_2 L(u)$ , so

$$\left| (f_1 - f_1^N)_u^*(t) \log_2 \frac{1}{t} \right| \leq (f_1)_u^*(t) \log_2 \frac{1}{t} \in L^1(0, \infty).$$

Therefore, by the dominated convergence theorem,

$$\|f_1 - f_1^N\|_{L\log_2 L(u)} = \int_0^{\infty} (f_1 - f_1^N)_u^*(t) \log_2 \frac{1}{t} dt \rightarrow 0,$$

as  $N \rightarrow \infty$ , so we finish the proof.  $\square$

#### 4. EXAMPLES AND APPLICATIONS

**4.1. Composition of Rubio de Francia operators.** Many times, we are interested in operators  $T$  that can be expressed as a composition of other (simpler) ones, say  $T \approx T_1 \circ T_2$ . It is clear that if we have boundedness information for  $T_1$  and  $T_2$ , and it can be put together as a composition, then we can draw conclusions for the original operator  $T$ . We will exemplify this with Calderón-Zygmund operators, although the same argument can be carried out with other operators for which their strong and weak-type  $(p, p)$  boundedness constants are known. We will also present an application to the composition of commutators of general linear operators.

A Calderón-Zygmund operator is an  $L^2(\mathbb{R}^n)$  bounded integral operator  $T$  whose kernel satisfies certain standard growth and smoothness conditions. This definition includes the

Hilbert, Beurling and Riesz transforms, among others, and it is well-known they are of strong-type  $(p, p)$  for every  $1 < p < \infty$  and every  $A_p$  weight. However, the sharp dependence of the boundedness constant on the weight was the result of several years of research culminating in [26]. In [27], the authors extended the result to maximal Calderón-Zygmund operators and included the weak-type case. If  $K$  is the kernel of a Calderón-Zygmund operator  $T$ , then we define

$$T_*f(x) = \sup_{0 < \varepsilon < \delta} \left| \int_{\varepsilon < |y| < \delta} K(x, y) f(y) dy \right|,$$

and the main result in [27] states that

- $\|T_*\|_{L^p(w) \rightarrow L^{p,\infty}(w)} \lesssim \|w\|_{A_p}, \quad 1 < p < 2,$
- $\|T_*\|_{L^p(w) \rightarrow L^p(w)} \lesssim \|w\|_{A_p}^{\frac{1}{p-1}}, \quad 1 < p < 2.$

Calderón-Zygmund operators and their maximal versions are known to be of weak-type  $(1,1)$  for  $A_1$  weights. However, if we consider a family  $\{T_i\}_{i=1}^k$ , with  $k \geq 2$ , this need not be true for the composition  $T_1 \circ \dots \circ T_k$ . By iteration of the previous estimates, we have that

- $\|T_{1,*} \circ \dots \circ T_{k,*}\|_{L^p(w) \rightarrow L^{p,\infty}(w)} \lesssim \|w\|_{A_p}^{1 + \frac{k-1}{p-1}}, \quad 1 < p < 2,$
- $\|T_{1,*} \circ \dots \circ T_{k,*}\|_{L^p(w) \rightarrow L^p(w)} \lesssim \|w\|_{A_p}^{\frac{k}{p-1}}, \quad 1 < p < 2.$

Using Corollary 2.7 on the weak-type estimate, we conclude the following:

**Corollary 4.1.** *Given a family of Calderón-Zygmund operators  $\{T_i\}_{i=1}^k$ , with  $k \geq 2$ , and  $u \in A_1$ , it holds that, for every  $\varepsilon > 0$ ,*

$$T_{1,*} \circ \dots \circ T_{k,*} : L(\log L)^{k-1+\varepsilon} \log_3 L(u) \longrightarrow R_{k-1+\varepsilon}(u).$$

Notice that if we had used the strong-type estimate with Corollary 2.2, we would have gotten boundedness on  $L(\log L)^k(u)$ , which is a smaller space.

Our next example will seek endpoint estimates for the composition of commutators. First, recall that a locally integrable function  $b : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be in  $BMO$  if

$$\|b\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy < \infty,$$

where  $b_Q = \frac{1}{|Q|} \int_Q b$  is the average of  $b$  on the cube  $Q$ . Given an operator  $T$  and a  $BMO$  function  $b$ , we define the commutator

$$[b, T]f = bTf - T(bf).$$

We can also define the  $k$ -th order commutator as  $T_b^k = [b, T_b^{k-1}]$  for every  $k \geq 1$ , being  $T_b^0 = T$ . The special case when  $T$  is a Calderón-Zygmund operator was first considered in [15], and in [34] it was shown that  $[b, T]$  is not of weak-type  $(1,1)$ . This motivates the study of endpoint estimates for commutators. In our case, however, we will deal with general linear operators  $T$  and their commutators with  $BMO$  functions, in the spirit of [14]. More precisely, we will make use of the following result, that can be found in [14, Corollary 3.2] for  $p_0 = 2$  and extended to  $1 < p_0 < \infty$  by obvious modifications.

**Theorem 4.2.** *Let  $T$  be a linear operator,  $1 < p_0 < \infty$  and  $b \in BMO$ . If*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded for every  $w \in A_{p_0}$  with constant  $C\|w\|_{A_{p_0}}^s$ , then, for every  $k \geq 1$  and every  $w \in A_{p_0}$ ,

$$T_b^k : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded with constant less than or equal to  $C_k \|b\|_{BMO}^k \|w\|_{A_{p_0}}^{k \max\{1, \frac{1}{p_0-1}\} + s}$ .

In view of this behavior, a direct application of Corollary 2.2 produces the following weighted endpoint result for the  $k$ -th order commutator and  $A_1$  weights.

**Corollary 4.3.** *Let  $T$  be a linear operator,  $1 < p_0 < \infty$  and  $b \in BMO$ . If*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded for every  $w \in A_{p_0}$  with constant  $C\|w\|_{A_{p_0}}^s$ , then, for every  $k \geq 1$  and every  $u \in A_1$ ,

$$T_b^k : L(\log L)^{k \max\{p_0-1, 1\} + s(p_0-1)}(u) \longrightarrow E_{k \max\{p_0-1, 1\} + s(p_0-1)}(u)$$

is also bounded with constant less than or equal to  $C_k \|b\|_{BMO}^k \|u\|_{A_1}^{k \max\{p_0, p_0'\} + sp_0}$ .

**4.2. The Carleson maximal operator.** Our next application will yield an  $A_1$  weighted endpoint estimate for the Carleson operator, defined by

$$\mathcal{C}f(x) = \sup_{a \in \mathbb{R}} \left| \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{e^{2\pi i a y} f(y)}{x-y} dy \right|.$$

Our argument will be based on a restricted weak-type  $(p, p)$  estimate for  $\mathcal{C}$  with respect to  $A_1$  weights.

**Proposition 4.4.** *Given  $1 < p \leq 2$  and  $u \in A_1$ , it holds that, for every measurable set  $E \subseteq \mathbb{R}$ ,*

$$\|\mathcal{C}\chi_E\|_{L^{p,\infty}(u)} \leq \frac{C_u}{p-1} u(E)^{1/p},$$

for some  $C_u > 1$  depending on the weight.

*Proof.* In [25], the authors prove the following good- $\lambda$  inequality for  $\mathcal{C}$ :

$$|\{x \in I_j : \mathcal{C}f(x) > 3\lambda, M_p f(x) \leq \gamma\lambda\}| \leq C \|\mathcal{C}\|_{L^p(\mathbb{R}) \rightarrow L^{p,\infty}(\mathbb{R})}^p \gamma^p |I_j|,$$

for every  $\gamma, \lambda > 0$ ,  $p > 1$ , and with  $\{I_j\}_j$  being a family of disjoint open intervals such that

$$\{\mathcal{C}f > \lambda\} = \bigcup_j I_j.$$

As usual, the operator  $M_p$  is defined by  $M_p f = (M(f^p))^{1/p}$ . If we only consider characteristic functions  $f = \chi_E$ , we can go over the proof in [25] and check that we can replace  $\|\mathcal{C}\|_{L^p(\mathbb{R}) \rightarrow L^{p,\infty}(\mathbb{R})}$  by  $\frac{C}{p-1}$ , using the well-known estimate by R. Hunt [24]:

$$(4.1) \quad \|\mathcal{C}\chi_E\|_{L^{p,\infty}(\mathbb{R})} \lesssim \frac{|E|^{1/p}}{p-1}, \quad 1 < p \leq 2.$$

Therefore, we get that, for every measurable set  $E \subseteq \mathbb{R}$ ,

$$|\{x \in I_j : \mathcal{C}\chi_E(x) > 3\lambda, M\chi_E(x) \leq \gamma^p \lambda^p\}| \leq \frac{C\gamma^p}{(p-1)^p} |I_j|.$$

Now, define

$$B = \{\mathcal{C}\chi_E > 3\lambda, M\chi_E \leq \gamma^p \lambda^p\},$$

and pick  $\gamma > 0$  in such a way that  $\frac{C\gamma^p}{(p-1)^p} = \varepsilon$ , for some  $\varepsilon > 0$  to be chosen later. Then, we can just write

$$|I_j \cap B| \leq \varepsilon |I_j|.$$

Using the sharp Reverse Hölder property of  $A_1$  weights (see [35]), we know that, for  $\delta = 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}$ ,

$$u(I_j \cap B) \leq u^\delta(I_j)^{1/\delta} |I_j \cap B|^{1/\delta'} \leq 2u(I_j) |I_j|^{-1/\delta'} |I_j \cap B|^{1/\delta'} \leq 2\varepsilon^{1/\delta'} u(I_j).$$

With this estimate and recalling the definition of the intervals  $I_j$ , we have that

$$\begin{aligned} u(\{\mathcal{C}\chi_E > 3\lambda\}) &\leq \sum_j u(I_j \cap B) + u(\{M\chi_E > \gamma^p \lambda^p\}) \\ &\leq 2\varepsilon^{1/\delta'} u(\{\mathcal{C}\chi_E > \lambda\}) + u(\{M\chi_E > \gamma^p \lambda^p\}), \end{aligned}$$

and hence,

$$\|\mathcal{C}\chi_E\|_{L^{p,\infty}(u)}^p \leq 3^p 2\varepsilon^{1/\delta'} \|\mathcal{C}\chi_E\|_{L^{p,\infty}(u)}^p + 3^p \gamma^{-p} \|M\chi_E\|_{L^1(u)}.$$

Now we choose  $\varepsilon > 0$  such that  $3^p 2\varepsilon^{1/\delta'} = 1/2$ , and using (2.2),

$$\|\mathcal{C}\chi_E\|_{L^{p,\infty}(u)}^p \leq 2 \cdot 3^p \gamma^{-p} \|u\|_{A_1} u(E).$$

To conclude, we only need to recall the value of  $\gamma$ ,  $\varepsilon$  and  $\delta$  to write

$$\|\mathcal{C}\chi_E\|_{L^{p,\infty}(u)} \leq \frac{C^{1/p} 2^{1/p} \cdot 3(4 \cdot 3^p)^{\frac{1+2^{n+1}\|u\|_{A_1}}{p}} \|u\|_{A_1}^{1/p}}{p-1} u(E)^{1/p} = \frac{C_u}{p-1} u(E)^{1/p}.$$

□

**Theorem 4.5.** *For every  $u \in A_1$ , we have that*

$$\mathcal{C} : L \log L \log_3 L(u) \longrightarrow R_1(u)$$

*is bounded.*

*Proof.* First, notice that the Carleson operator  $\mathcal{C}$  is  $(\varepsilon, \delta)$ -atomic approximable. The easiest way to check this is to recall that

$$\mathcal{C}f(x) \approx f(x) + \sup_{R \in \mathbb{Q}} \left| \int_{-R}^R \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \right| = f(x) + \sup_{R \in \mathbb{Q}} |S_R f(x)|,$$

where  $S_R$  is of convolution type with kernel in  $L^2(\mathbb{R})$ , for every  $R \in \mathbb{Q}$ . Hence, by (3.3),  $\{S_R\}_{R \in \mathbb{Q}}$  is a sequence of  $(\varepsilon, \delta)$ -atomic operators and we prove our claim. The result now follows by Remarks 2.9 and 2.14 together with the estimate in Proposition 4.4. □

**Remark 4.6.** *Notice that, in Proposition 4.4, we could have worked with the good- $\lambda$  inequality for general functions (and hence, with the norm  $\|\mathcal{C}\|_{L^p(\mathbb{R}) \rightarrow L^{p,\infty}(\mathbb{R})}$ ). In this case, we would have gotten weak-type  $(p, p)$  estimates instead of restricted weak-type ones. However, the fact that  $\mathcal{C}$  is  $(\varepsilon, \delta)$ -atomic approximable and that it satisfies (4.1) with a good constant, makes that, in terms of extrapolation, it is more interesting to deal with characteristic functions.*

*Theorem 4.5 extends Antonov's endpoint estimate [1] for  $\mathcal{C}$  and shows that the Fourier integral on  $\mathbb{R}$  is pointwise convergent for every function  $f \in L \log L \log_3 L(u)$  with  $u \in A_1$ .*

**4.3. Maximally modulated singular integrals.** In view of the previous subsection, we can prove the following general result that only relies on a certain good- $\lambda$  inequality. As we shall see, a wide class of operators called maximally modulated singular integrals will fall within the scope of this result (see [23, 19]).

**Theorem 4.7.** *Assume that  $T$  is a sublinear operator such that, for an increasing function  $\psi$  on  $[1, \infty)$ , it satisfies*

$$|\{x \in Q_j : Tf(x) > 3\lambda, M_p f(x) \leq \gamma\lambda\}| \leq \psi\left(\frac{1}{p-1}\right)^p \gamma^p |Q_j|,$$

for every  $\gamma, \lambda > 0$ ,  $1 < p \leq p_0$ , and with  $\{Q_j\}_j$  being a family of disjoint open cubes covering  $\{Tf > \lambda\}$ . Alternatively, we can assume that the same holds only on characteristic functions  $f = \chi_E$  with a function  $\psi_r$  instead of  $\psi$ . Then, when  $u \in A_1$ ,

- $T : L^p(u) \rightarrow L^{p,\infty}(u)$  is bounded with constant controlled by  $C_u \psi\left(\frac{1}{p-1}\right)$ .
- It holds that

$$\|T\chi_E\|_{L^{p,\infty}(u)} \leq C_u \psi_r\left(\frac{1}{p-1}\right) u(E)^{1/p}.$$

Naturally, depending on the expression of  $\psi$  (or  $\psi_r$ ), and whether  $T$  is  $(\varepsilon, \delta)$ -atomic or not, one can try to extrapolate these estimates for particular examples of  $T$ , as we did in Theorem 4.5 for the Carleson operator  $\mathcal{C}$  with  $\psi_r = \text{Id}$ . As we anticipated, other examples to which we can apply Theorem 4.7 are the so-called maximally modulated Calderón-Zygmund (maximal) operators. Following the presentation in [23], we recall that given a standard Calderón-Zygmund operator  $T$  with kernel  $K$  and a family  $\Phi = \{\phi_a\}_{a \in A}$  of measurable real-valued functions indexed by an arbitrary set  $A$ , we can define the maximally modulated  $T$  with respect to  $\Phi$ :

$$T^\Phi f(x) = \sup_{a \in A} |T(e^{2\pi i \phi_a(\cdot)} f)(x)|.$$

This definition is motivated by the Carleson operator, for which  $T$  is the Hilbert transform and  $\Phi$  is given by  $\phi_a(y) = ay$ , for every  $a \in \mathbb{R}$ . Also, mimicking what we do with singular integrals, we can define the maximal version of  $T^\Phi$  by

$$T_*^\Phi f(x) = \sup_{\varepsilon > 0} \sup_{a \in A} |T_\varepsilon(e^{2\pi i \phi_a(\cdot)} f)(x)|,$$

where  $T_\varepsilon$  is the truncated operator defined by

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x,y) f(y) dy.$$

In the proof of the main result in [23], if we keep track of the constants, the authors show the following:

**Theorem 4.8.** *Let  $T$  be a Calderón-Zygmund operator and let  $\Phi$  be a family of measurable real-valued functions. Take  $\mathcal{T} \in \{T^\Phi, T_*^\Phi\}$ . Assume that  $\mathcal{T}$  maps  $L^p(\mathbb{R}^n)$  into  $L^{p,\infty}(\mathbb{R}^n)$  for*

$p > 1$  with norm  $\|\mathcal{T}\|_{L^p(\mathbb{R}^n) \rightarrow L^{p,\infty}(\mathbb{R}^n)}$ . Then,  $\mathcal{T}$  is under the hypotheses of Theorem 4.7 with

$$\psi\left(\frac{1}{p-1}\right) = C\|\mathcal{T}\|_{L^p(\mathbb{R}^n) \rightarrow L^{p,\infty}(\mathbb{R}^n)}.$$

Alternatively, if we have an estimate on characteristic functions  $\|\mathcal{T}\chi_E\|_{L^{p,\infty}(\mathbb{R}^n)} \leq C_p|E|^{1/p}$ , then  $\mathcal{T}$  is under the hypotheses of Theorem 4.7 with

$$\psi_r\left(\frac{1}{p-1}\right) = C_p.$$

**Remark 4.9.** Since  $T$  is a Calderón-Zygmund operator, we know that it satisfies Cotlar's inequality and one can readily show that  $T_*^\Phi f(x) \lesssim Mf(x) + M(T^\Phi f)(x)$ . With this, together with the bounds  $\|M\|_{L^{p,\infty}(\mathbb{R}^n) \rightarrow L^{p,\infty}(\mathbb{R})} \lesssim \frac{1}{p-1}$  and  $\|M\|_{L^{p,\infty}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R})} \leq C$  for  $p > 1$ , we get that

$$\|T_*^\Phi\|_{L^p(\mathbb{R}^n) \rightarrow L^{p,\infty}(\mathbb{R}^n)} \lesssim \frac{\|T^\Phi\|_{L^p(\mathbb{R}^n) \rightarrow L^{p,\infty}(\mathbb{R}^n)}}{p-1}.$$

Therefore, one can always write a good- $\lambda$  inequality for  $T_*^\Phi$  in terms of  $\|T^\Phi\|_{L^p(\mathbb{R}^n) \rightarrow L^{p,\infty}(\mathbb{R}^n)}$ .

The combination of Theorems 4.7 and 4.8 is similar to the results presented in [19], where the authors study weighted strong-type  $(p, p)$  estimates for maximally modulated singular integrals  $T^\Phi$  that satisfy an *a priori* weak-type  $(p, p)$  inequality without weights. In the same paper, the authors also show that, for  $1 < p \leq 2$ ,

$$(4.2) \quad \|\mathcal{C}\|_{L^p(\mathbb{R}) \rightarrow L^{p,\infty}(\mathbb{R})} \lesssim \frac{\log_2\left(\frac{1}{p-1}\right)}{p-1},$$

and

$$(4.3) \quad \|\mathcal{C}_{\text{lac}}\|_{L^p(\mathbb{R}) \rightarrow L^{p,\infty}(\mathbb{R})} \lesssim \log_1\left(\frac{1}{p-1}\right).$$

Here  $\mathcal{C}_{\text{lac}}$  is the lacunary version of  $\mathcal{C}$ , defined as a maximally modulated Hilbert transform  $H^\Phi$  with  $\Phi = \{ay\}_{a \in A}$ , where the index set  $A \subseteq \mathbb{R}$  is lacunary, in the sense that

$$\inf_{a \neq a' \in A} \frac{|a - a'|}{|a|} = C > 0.$$

As we pointed out in Remark 4.6, the use of (4.2) to obtain weighted  $(p, p)$  results and then extrapolate does not offer an improvement of Theorem 4.5. However, let us see what we can obtain for the lacunary Carleson operator:

**Corollary 4.10.** For every  $1 < p \leq 2$  and  $u \in A_1$ ,

$$\|\mathcal{C}_{\text{lac}}\|_{L^p(u) \rightarrow L^{p,\infty}(u)} \leq C_u \log_1\left(\frac{1}{p-1}\right).$$

In particular,

$$\mathcal{C}_{\text{lac}} : L \log_2 L \log_4 L(u) \longrightarrow R_1(u).$$

*Proof.* The weak-type  $(p, p)$  estimates come from (4.3) together with Theorems 4.7 and 4.8. The endpoint estimate can be obtained by a suitable modification of Antonov's Theorem 2.6 so that it admits the logarithmic blow-up of the constant.  $\square$

In this lacunary case, the current best result for the Carleson operator  $\mathcal{C}_{\text{lac}}$  is boundedness on  $L \log_2 L \log_4 L(\mathbb{T})$  (see [19, 30]), and in fact, it has been recently showed [31] that this is the largest Lorentz space over  $\mathbb{T}$  on which  $\mathcal{C}_{\text{lac}}$  can be bounded. Hence, what we obtain is an analogue of this endpoint result on  $\mathbb{R}$  and with respect to  $A_1$  weights.

**4.4. The operator  $A$ .** In this subsection, we shall deal with the operator

$$Af(x) = \left\| \frac{f(x+y)}{y} \right\|_{L^{1,\infty}(\mathbb{R})},$$

We show a weighted endpoint result for  $A$  in the spirit of Theorem 3.3. In [18], the authors prove that a version of  $A$  defined on the probability space  $([0, 1], dx)$  is bounded on  $L \log_2 L([0, 1])$ , and show that in the scale of Orlicz spaces,  $L \log_3 L([0, 1])$  would be the best possible result. We will show the following:

**Theorem 4.11.** *For every  $u \in A_1$ , it holds that*

$$A : L \log_2 L(u) \longrightarrow L_{\text{loc}}^{1,\infty}(u).$$

*Proof.* Since  $A\chi_E \leq M\chi_E$ , the proof is essentially an application of Theorem 3.3. However, we need to make sure that the lack of sublinearity of  $A$  is not a problem. The three properties that will replace the missing hypothesis are:

- $A$  is quasilinear:  $A(f+g) \leq C(Af+Ag)$ ,
- $A$  is monotone:  $|f| \leq |g| \Rightarrow Af \leq Ag$ ,
- $A$  is sublinear on functions with disjoint support.

Now, let us examine the proof of Theorem 3.3. Whenever we use sublinearity on a sum of two functions, the quasilinearity of  $A$  suffices. The only step where we need it for an arbitrary number of terms is in (3.5), and here quasilinearity is not enough. To get around this problem, we replace (3.5) by the standard dyadic decomposition

$$(4.4) \quad f \approx \sum_{k \in \mathbb{Z}} 2^k \chi_{\{2^k < f \leq 2^{k+1}\}}.$$

Since this is an equivalence instead of an equality, we require monotonicity to pass the operator inside the sum. Once this is done, we use that the pieces of (4.4) have disjoint support and  $A$  behaves as if it were sublinear. For the decomposition in (4.4) to be finite we only have to start with bounded functions and at the end, add a density argument.  $\square$

## 5. APPENDIX

*Proof of Lemma 2.20.* The proof of this result can be found, for instance, in [5, Theorem 5.3.2], but we need to see how the constant behaves and this is not included in classical books. We will proceed as in [10, Lemma 3.10]. By the real interpolation  $K$ -method (see [4, Chapter 5]), we have that

$$T : (L^{q_0, s_0}(u), L^{q_1, s_1}(u))_{\theta, \infty} \longrightarrow (L^{q_0, \infty}(u), L^{q_1, \infty}(u))_{\theta, \infty},$$

with constant less than or equal to  $M_0^{1-\theta} M_1^\theta$ , where

$$(A_1, A_2)_{\theta, \infty} = \left\{ f \in A_1 + A_2 : \sup_{t>0} t^{-\theta} K(t, f; A_1, A_2) < \infty \right\},$$



and

$$K(t, f; A_1, A_2) = \inf \{ \|f_0\|_{A_1} + t\|f_1\|_{A_2} : f = f_0 + f_1, f_0 \in A_1, f_1 \in A_2 \}.$$

Therefore, it is enough to show that:

- (i)  $\|f\|_{L^{q,\infty}(u)} \leq 2\|f\|_{(L^{q_0,\infty}(u), L^{q_1,\infty}(u))_{\theta,\infty}},$
- (ii)  $\|f\|_{(L^{q_0,s_0}(u), L^{q_1,s_1}(u))_{\theta,\infty}} \leq B\|f\|_{L^{q,\infty}(u)}.$

The proof of (i) goes as follows: define  $\gamma := \frac{q_0 q_1}{q_1 - q_0}$ , fix  $t > 0$  and let  $f = f_0 + f_1$  be a decomposition of  $f$  in  $L^{q_0,\infty}(u) + L^{q_1,\infty}(u)$ . Then,

$$\begin{aligned} \sup_{y \leq t^\gamma} y^{1/q_0} f_u^*(y) &\leq \sup_{y \leq t^\gamma} y^{1/q_0} \left( (f_0)_u^* \left( \frac{y}{2} \right) + (f_1)_u^* \left( \frac{y}{2} \right) \right) \\ &\leq \sup_{y \leq t^\gamma} 2^{1/q_0} \|f_0\|_{L^{q_0,\infty}(u)} + y^{\frac{1}{q_0} - \frac{1}{q_1}} 2^{1/q_1} \|f_1\|_{L^{q_1,\infty}(u)} \\ &\leq 2(\|f_0\|_{L^{q_0,\infty}(u)} + t\|f_1\|_{L^{q_1,\infty}(u)}). \end{aligned}$$

Taking infimum over all possible decompositions of  $f$ , we conclude that

$$\sup_{y \leq t^\gamma} y^{1/q_0} f_u^*(y) \leq 2K(t, f; L^{q_0,\infty}(u), L^{q_1,\infty}(u)),$$

and with this estimate,

$$\begin{aligned} 2\|f\|_{(L^{q_0,\infty}(u), L^{q_1,\infty}(u))_{\theta,\infty}} &= \sup_{t>0} 2t^{-\theta} K(t, f; L^{q_0,\infty}(u), L^{q_1,\infty}(u)) \\ &\geq \sup_{t>0} \sup_{y \leq t^\gamma} t^{-\theta} y^{1/q_0} f_u^*(y) = \sup_{y>0} y^{1/q_0} f_u^*(y) \sup_{t \geq y^{1/\gamma}} t^{-\theta} \\ &= \sup_{y>0} y^{\frac{-\theta}{\gamma} + \frac{1}{q_0}} f_u^*(y) = \|f\|_{L^{q,\infty}(u)}. \end{aligned}$$

For (ii), let  $f \in L^{q,\infty}(u)$  and  $\gamma$  as before. For every  $t > 0$ , we write  $f = f_0 + f_1$  with

$$f_0 = f \chi_{\{|f| > f_u^*(t^\gamma)\}} \quad \text{and} \quad f_1 = f \chi_{\{|f| \leq f_u^*(t^\gamma)\}}.$$

Now,

$$\begin{aligned} \|f_0\|_{L^{q_0,s_0}(u)} &\leq \left( \int_0^{t^\gamma} (f_u^*(y) y^{1/q})^{s_0} y^{\frac{s_0}{q_0} - \frac{s_0}{q} - 1} dy \right)^{1/s_0} \leq \|f\|_{L^{q,\infty}(u)} \frac{t^{\gamma(\frac{1}{q_0} - \frac{1}{q})}}{\left( \frac{s_0}{q_0} - \frac{s_0}{q} \right)^{1/s_0}} \\ &= t^\theta \left( \frac{q_0 q}{s_0(q - q_0)} \right)^{1/s_0} \|f\|_{L^{q,\infty}(u)}, \end{aligned}$$

by the definition of  $\gamma$  and  $\theta = \frac{q_0 q_1 - q q_1}{q q_0 - q q_1}$ . Also,

$$\|f_1\|_{L^{q_1,s_1}(u)} \leq f_u^*(t^\gamma) \left( \int_0^{t^\gamma} y^{\frac{s_1}{q_1} - 1} dy \right)^{1/s_1} + \left( \int_{t^\gamma}^\infty f_u^*(y)^{s_1} y^{\frac{s_1}{q_1} - 1} dy \right)^{1/s_1}.$$

For the first term, we multiply and divide by  $t^{\gamma/q}$ , compute the integral and the bound we get is

$$t^{\theta-1} \left( \frac{q_1}{s_1} \right)^{1/s_1} \|f\|_{L^{q,\infty}(u)}.$$

For the second term, we proceed exactly as for  $\|f_0\|_{L^{q_0, s_0}(u)}$  and control it by

$$t^{\theta-1} \left( \frac{q_1 q}{s_1 (q_1 - q)} \right)^{1/s_1} \|f\|_{L^{q, \infty}(u)}.$$

Bringing the estimates together, we conclude that

$$\begin{aligned} \|f\|_{(L^{q_0, s_0}(u), L^{q_1, s_1}(u))_{\theta, \infty}} &= \sup_{t>0} t^{-\theta} K(t, f; L^{q_0, s_0}(u), L^{q_1, s_1}(u)) \\ &\leq \sup_{t>0} t^{-\theta} (\|f_0\|_{L^{q_0, s_0}(u)} + t \|f_1\|_{L^{q_1, s_1}(u)}) \leq B \|f\|_{L^{q, \infty}(u)}. \end{aligned}$$

□

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