MULTI-PARAMETER EXTENSIONS OF A THEOREM OF PICHORIDES

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Abstract. Extending work of Pichorides and Zygmund to the d-dimensional setting, we show that the supremum of L^p-norms of the Littlewood-Paley square function over the unit ball of the analytic Hardy spaces $H^p_A(\mathbb{T}^d)$ blows up like $(p-1)^{-d}$ as $p \to 1^+$. Furthermore, we obtain an $L \log^d L$-estimate for square functions on $H^1_A(\mathbb{T}^d)$. Euclidean variants of Pichorides's theorem are also obtained.

1. Introduction

Given a trigonometric polynomial f on $\mathbb{T}$, we define the classical Littlewood-Paley square function $S_{\mathbb{T}}(f)$ of f by

$$S_{\mathbb{T}}(f)(x) = \Big(\sum_{k \in \mathbb{Z}} |\Delta_k(f)(x)|^2 \Big)^{1/2},$$

where for $k \in \mathbb{N}$, we set

$$\Delta_k(f)(x) = \sum_{n=2^{k-1}}^{2^k-1} \widehat{f}(n) e^{i2\pi nx} \quad \text{and} \quad \Delta_{-k}(f)(x) = \sum_{n=-2^k+1}^{-2^{k-1}} \widehat{f}(n) e^{i2\pi nx}$$

and for $k = 0$ we take $\Delta_0(f)(x) = \widehat{f}(0)$.

A classical theorem of J.E. Littlewood and R.E.A.C. Paley asserts that for every $1 < p < \infty$ there exists a constant $B_p > 0$ such that

$$\|S_{\mathbb{T}}(f)\|_{L^p(\mathbb{T})} \leqslant B_p \|f\|_{L^p(\mathbb{T})} \tag{1}$$

for every trigonometric polynomial f on $\mathbb{T}$, see, e.g., [5] or [20].

The operator $S_{\mathbb{T}}$ is not bounded on $L^1(\mathbb{T})$, and hence, the constant B_p in (1) blows up as $p \to 1^+$. In [2], J. Bourgain obtained the sharp estimate

$$B_p \sim (p-1)^{-3/2} \quad \text{as} \quad p \to 1^+. \tag{2}$$

For certain subspaces of $L^p(\mathbb{T})$, however, one might hope for better bounds. In [12], S. Pichorides showed that for the analytic Hardy spaces $H^p_A(\mathbb{T})$ $(1 < p \leqslant 2)$, we have

$$\sup_{\substack{\|f\|_{L^p(\mathbb{T})} \leqslant 1 \\ f \in H^p_A(\mathbb{T})}} \|S_{\mathbb{T}}(f)\|_{L^p(\mathbb{T})} \sim (p-1)^{-1} \quad \text{as} \quad p \to 1^+. \tag{3}$$

Higher-dimensional extensions of Bourgain's result (2) were obtained by the first author in [1]. In particular, given a dimension $d \in \mathbb{N}$, if f is a trigonometric polynomial on $\mathbb{T}^d$, we define its d-parameter Littlewood-Paley square function by

$$S_{\mathbb{T}^d}(f)(x) = \Big(\sum_{k_1, \cdots, k_d \in \mathbb{Z}} |\Delta_{k_1, \cdots, k_d}(f)(x)|^2 \Big)^{1/2},$$

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where for $k_1, \cdots, k_d \in \mathbb{Z}$ we use the notation $\Delta_{k_1,\cdots,k_d}(f) = \Delta_{k_1} \otimes \cdots \otimes \Delta_{k_d}(f)$, where the direct product notation indicates the operator Δ_{k_j} in the j-th position acting on the j-th variable. As in the one-dimensional case, for every $1 < p < \infty$, there is a positive constant $B_p(d)$ such that

$$\|S_{\mathbb{T}^d}(f)\|_{L^p(\mathbb{T}^d)} \leqslant B_p(d)\|f\|_{L^p(\mathbb{T}^d)}$$

for each trigonometric polynomial f on $\mathbb{T}^d$. It is shown in [1] that

$$B_p(d) \sim_d (p-1)^{-3d/2} \quad \text{as} \quad p \to 1^+. \tag{4}$$

A natural question in this context is whether one has an improvement on the limiting behaviour of $B_p(d)$ as $p \to 1^+$ when restricting to the analytic Hardy spaces $H^p_A(\mathbb{T}^d)$. In other words, one is led to ask whether the aforementioned theorem of Pichorides can be extended to the polydisc. However, the proof given in [12] relies on factorisation of Hardy spaces, and it is known, see for instance [14, Chapter 5] and [13], that canonical factorisation fails in higher dimensions.

In this note we obtain an extension of (3) to the polydisc as a consequence of a more general result involving tensor products of Marcinkiewicz multiplier operators on $\mathbb{T}^d$. Recall that, in the periodic setting, a multiplier operator T_m associated to a function $m \in \ell^\infty(\mathbb{Z})$ is said to be a Marcinkiewicz multiplier operator on the torus if m satisfies

$$B_m = \sup_{k\in\mathbb{N}} \Big[\sum_{n=2^k-1}^{2^{k+1}} |m(n+1) - m(n)| + \sum_{n=-2^{k+1}}^{-2^k+1} |m(n+1) - m(n)| \Big] < \infty. \tag{5}$$

Our main result in this paper is the following theorem.

Theorem 1. *Let $d \in \mathbb{N}$ be a given dimension. If T_{m_j} is a Marcinkiewicz multiplier operator on $\mathbb{T}$ $(j = 1, \cdots, d)$, then for every $f \in H^p_A(\mathbb{T}^d)$ one has*

$$\|(T_{m_1} \otimes \cdots \otimes T_{m_d})(f)\|_{L^p(\mathbb{T}^d)} \lesssim_{C_{m_1},\cdots,C_{m_d}} (p-1)^{-d}\|f\|_{L^p(\mathbb{T}^d)}$$

as $p \to 1^+$, where $C_{m_j} = \|m_j\|_{\ell^\infty(\mathbb{Z})} + B_{m_j}$, B_{m_j} being as in (5), $j = 1, \cdots, d$.

To prove Theorem 1, we use a theorem of T. Tao and J. Wright [16] on the endpoint mapping properties of Marcinkiewicz multiplier operators on the line, transferred to the periodic setting, with a variant of Marcinkiewicz interpolation for Hardy spaces which is due to S. Kislyakov and Q. Xu [8].

Since for every choice of signs the randomised version $\sum_{k\in\mathbb{Z}} \pm\Delta_k$ of $S_{\mathbb{T}}$ is a Marcinkiewicz multiplier operator on the torus with corresponding constant $B_m \leqslant 2$, Theorem 1 and Khintchine's inequality yield the following d-parameter extension of Pichorides's theorem (3).

Corollary 2. *Given $d \in \mathbb{N}$, if $S_{\mathbb{T}^d}$ denotes the d-parameter Littlewood-Paley square function, then one has*

$$\sup_{\substack{\|f\|_{L^p(\mathbb{T}^d)}\leqslant 1 \\ f\in H^p_A(\mathbb{T}^d)}} \|S_{\mathbb{T}^d}(f)\|_{L^p(\mathbb{T}^d)} \sim_d (p-1)^{-d} \quad \textit{as} \quad p \to 1^+.$$

The present paper is organised as follows: In the next section we set down notation and provide some background, and in Section 3 we prove our main results. In Section 4 we give another application of Theorem 1 that extends a well-known inequality due to A. Zygmund [19, Theorem 8]. In the last section we obtain a Euclidean version of Theorem 1 by using the aforementioned theorem of Tao and Wright [16] combined with a theorem of Peter Jones [7] on a Marcinkiewicz-type decomposition for analytic Hardy spaces over the real line.

2. Preliminaries

2.1. **Notation.** We denote the set of natural numbers by $\mathbb{N}$, by $\mathbb{N}_0$ the set of non-negative integers and by $\mathbb{Z}$ the set of integers.

Let f be a function of d-variables. Fixing the first $d-1$ variables $(x_1, \cdots, x_{d-1})$, we write $f(x_1, \cdots, x_d) = f_{(x_1,\cdots,x_{d-1})}(x_d)$. The sequence of the Fourier coefficients of a function $f \in L^1(\mathbb{T}^d)$ will be denoted by $\hat{f}$.

Given a function $m \in L^\infty(\mathbb{R}^d)$, we denote by T_m the multiplier operator corresponding to m, initially defined on $L^2(\mathbb{R}^d)$, by $(T_m(f))\hat{}\,(\xi) = m(\xi)\widehat{f}(\xi)$, $\xi \in \mathbb{R}^d$. Given $\mu \in \ell^\infty(\mathbb{Z}^d)$, one defines (initially on $L^2(\mathbb{T}^d)$) the corresponding periodic multiplier operator T_μ in an analogous way.

If λ is a continuous and bounded function on the real line and T_λ is as above, $T_{\lambda|_\mathbb{Z}}$ denotes the periodic multiplier operator such that $T_{\lambda|_\mathbb{Z}}(f)(x) = \sum_{n\in\mathbb{Z}} \lambda(n)\widehat{f}(n)e^{i2\pi nx}$ $(x \in \mathbb{T})$ for every trigonometric polynomial f on $\mathbb{T}$.

Given two positive quantities X and Y and a parameter α, we write $X \lesssim_\alpha Y$ (or simply $X \lesssim Y$) whenever there exists a constant $C_\alpha > 0$ depending on α so that $X \leqslant C_\alpha Y$. If $X \lesssim_\alpha Y$ and $Y \lesssim_\alpha X$, we write $X \sim_\alpha Y$ (or simply $X \sim Y$).

2.2. **Hardy spaces and Orlicz spaces.** Let $d \in \mathbb{N}$. For $0 < p < \infty$, let $H^p_A(\mathbb{D}^d)$ denote the space of holomorphic functions F on $\mathbb{D}^d$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, such that

$$\|F\|^p_{H^p_A(\mathbb{D}^d)} = \sup_{0\leqslant r_1,\cdots,r_d<1} \int_{\mathbb{T}^d} |F(r_1e^{i2\pi x_1}, \cdots, r_de^{i2\pi x_d})|^p dx_1 \cdots dx_d < \infty.$$

For $p = \infty$, $H^\infty_A(\mathbb{D}^d)$ denotes the class of bounded holomorphic functions on $\mathbb{D}^d$. It is well-known that for $1 \leqslant p \leqslant \infty$, the limit f of $F \in H^p_A(\mathbb{D}^d)$ as we approach the distinguished boundary $\mathbb{T}^d$ of $\mathbb{D}^d$, namely

$$f(x_1, \cdots, x_d) = \lim_{r_1,\cdots,r_d\to 1^-} F(r_1e^{i2\pi x_1}, \cdots, r_de^{i2\pi x_d})$$

exists a.e. in $\mathbb{T}^d$ and $\|F\|_{H^p_A(\mathbb{D}^d)} = \|f\|_{L^p(\mathbb{T}^d)}$. For $1 \leqslant p \leqslant \infty$, we define the analytic Hardy space $H^p_A(\mathbb{T}^d)$ on the d-torus as the space of all functions in $L^p(\mathbb{T}^d)$ that are boundary values of functions in $H^p_A(\mathbb{D}^d)$. Moreover, it is a standard fact that $H^p_A(\mathbb{T}^d) = \{f \in L^p(\mathbb{T}^d)\colon\ \mathrm{supp}(\widehat{f}) \subset \mathbb{N}^d_0\}$. For $1 \leqslant p \leqslant \infty$, $(H^p_A(\mathbb{T}^d), \|\cdot\|_{L^p(\mathbb{T}^d)})$ is a Banach space and for $1 < p < \infty$ one has

$$\|f\|_{L^p(\mathbb{T}^d)} = \sup_{\substack{\|g\|_{L^q(\mathbb{T}^d)}=1 \\ g\in H^q_A(\mathbb{T}^d)}} \left| \int_{\mathbb{T}^d} f(x)\overline{g(x)}dx \right|,$$

where $p^{-1} + q^{-1} = 1$. Hardy spaces are discussed in Chapter 7 in [4], where the case $d = 1$ is treated, and in Chapter 3 of [14].

If $f \in L^1(\mathbb{T}^d)$ is such that $\mathrm{supp}(\widehat{f})$ is finite, then f is said to be a trigonometric polynomial on $\mathbb{T}^n$, and if moreover $\mathrm{supp}(\widehat{f}) \subset \mathbb{N}^d_0$, then f is said to be analytic. It is well-known [4, 14] that for $1 \leqslant p < \infty$, the class of trigonometric polynomials on $\mathbb{T}^d$ is a dense subspace of $L^p(\mathbb{T}^d)$ and analytic trigonometric polynomials on $\mathbb{T}^d$ are dense in $H^p_A(\mathbb{T}^d)$.

We define the real Hardy space $H^1(\mathbb{T})$ to be the space of all integrable functions $f \in L^1(\mathbb{T})$ such that $H_\mathbb{T}(f) \in L^1(\mathbb{T})$, where $H_\mathbb{T}(f)$ denotes the periodic Hilbert transform of f. One sets $\|f\|_{H^1(\mathbb{T})} = \|f\|_{L^1(\mathbb{T})} + \|H_\mathbb{T}(f)\|_{L^1(\mathbb{T})}$. Note that $H^1_A(\mathbb{T})$ can be regarded as a proper subspace of $H^1(\mathbb{T})$ and moreover, $\|f\|_{H^1(\mathbb{T})} = 2\|f\|_{L^1(\mathbb{T})}$ when $f \in H^1_A(\mathbb{T})$.

Given $d \in \mathbb{N}$, for $0 < p < \infty$, let $H^p_A((\mathbb{R}^2_+)^d)$ denote the space of holomorphic functions F on $(\mathbb{R}^2_+)^d$, where $\mathbb{R}^2_+ = \{x + iy \in \mathbb{C} : y > 0\}$, such that

$$\|F\|^p_{H^p_A((\mathbb{R}^2_+)^d)} = \sup_{y_1,\cdots,y_d>0} \int_{\mathbb{R}^d} |F(x_1 + iy_1, \cdots, x_d + iy_d)|^p dx_1 \cdots dx_d < \infty.$$

For $p = \infty$, $H^\infty_A((\mathbb{R}^2_+)^d)$ is defined as the space of bounded holomorphic functions in $(\mathbb{R}^2_+)^d$. For $1 \leqslant p \leqslant \infty$, for every $F \in H^p_A((\mathbb{R}^2_+)^d)$ its limit f as we approach the boundary $\mathbb{R}^d$, namely

$$f(x_1, \cdots, x_d) = \lim_{y_1,\cdots,y_d \to 0^+} F(x_1 + iy_1, \cdots, x_d + iy_d),$$

exists for a.e. $(x_1, \cdots, x_d) \in \mathbb{R}^d$ and, moreover, $\|F\|_{H^p_A((\mathbb{R}^2_+)^d)} = \|f\|_{L^p(\mathbb{R}^d)}$. Hence, as in the periodic setting, for $1 \leqslant p \leqslant \infty$ we may define the d-parameter analytic Hardy space $H^p_A(\mathbb{R}^d)$ to be the space of all functions in $L^p(\mathbb{R}^d)$ that are boundary values of functions in $H^p_A((\mathbb{R}^2_+)^d)$.

The real Hardy space $H^1(\mathbb{R})$ on the real line is defined as the space of all integrable functions f on $\mathbb{R}$ such that $H(f) \in L^1(\mathbb{R})$, where $H(f)$ is the Hilbert transform of f. Moreover, we set $\|f\|_{H^1(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})} + \|H(f)\|_{L^1(\mathbb{R})}$.

We shall also consider the standard Orlicz spaces $L \log^r L(\mathbb{T}^d)$ and $\exp L^{1/r}(\mathbb{T}^d)$. For $r > 0$, one may define $L \log^r L(\mathbb{T}^d)$ as the space of measurable functions f on $\mathbb{T}^d$ such that $\int_{\mathbb{T}^d} |f(x)| \log^r(1 + |f(x)|)dx < \infty$. For $r \geqslant 1$, we may equip $L \log^r L(\mathbb{T}^d)$ with a norm given by

$$\|f\|_{L \log^r L(\mathbb{T}^d)} = \inf \big\{\lambda > 0 : \int_{\mathbb{T}^d} \frac{|f(x)|}{\lambda} \log^r \big(1 + \frac{|f(x)|}{\lambda}\big) dx \leqslant 1\big\}.$$

Also, one may define

$$\|f\|_{\exp L^{1/r}(\mathbb{T}^d)} = \sup_{q \in \mathbb{N}_0} q^{-r} \|f\|_{L^q(\mathbb{T}^d)}.$$

It is well-known that the spaces $L \log^r L(\mathbb{T}^d)$ and $\exp L^{1/r}(\mathbb{T}^d)$ can be viewed as duals and in particular, one has

$$\left| \int_{\mathbb{T}^d} f(x) g(x) dx \right| \leqslant C_{r,d} \|f\|_{L \log^r L(\mathbb{T}^d)} \|g\|_{\exp L^{1/r}(\mathbb{T}^d)} \tag{6}$$

where $C_{r,d} > 0$ is a constant depending only on r, d. For more details on Orlicz spaces, we refer the reader to the books [9] and [20].

3. Proof of Theorem 1

Recall that a function $m \in L^\infty(\mathbb{R})$ is said to be a Marcinkiewicz multiplier on $\mathbb{R}$ if it is differentiable in every dyadic interval $\pm[2^k, 2^{k+1})$, $k \in \mathbb{Z}$ and

$$A_m = \sup_{k \in \mathbb{Z}} \big[\int_{[2^k, 2^{k+1})} |m'(\xi)| d\xi + \int_{(-2^{k+1}, -2^k]} |m'(\xi)| d\xi \big] < \infty \tag{7}$$

If $m \in L^\infty(\mathbb{R})$ satisfies (7), then thanks to a classical result of Marcinkiewicz[1], see e.g. [15], the corresponding multiplier operator T_m is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$. In [16], Tao and Wright showed that every Marcinkiewicz multiplier operator T_m is bounded from the real Hardy space $H^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$, namely

$$\|T_m(f)\|_{L^{1,\infty}(\mathbb{R})} \leqslant C_m \|f\|_{H^1(\mathbb{R})}, \tag{8}$$

[1]J. Marcinkiewicz originally proved the theorem in the periodic setting, see [10].

where the constant C_m depends only on $\|m\|_{L^\infty(\mathbb{R})} + A_m$, with A_m as in (7). For the sake of completeness, let us recall that $L^{1,\infty}(\mathcal{M})$ stands for the quasi-Banach space of measurable functions on a measure space $\mathcal{M}$ endowed with the quasinorm

$$\|f\|_{L^{1,\infty}(\mathcal{M})} := \sup_{t>0} t \int_{\{x\in\mathcal{M}:|f(x)|>t\}} dx.$$

Either by adapting the proof of Tao and Wright to the periodic setting or by using a transference argument, see Subsection 3.2, one deduces that every periodic Marcinkiewicz multiplier operator T_m satisfies

$$\|T_m(f)\|_{L^{1,\infty}(\mathbb{T})} \leqslant D_m \|f\|_{H^1(\mathbb{T})}, \tag{9}$$

where D_m depends on $\|m\|_{\ell^\infty(\mathbb{Z})} + B_m$, B_m being as in (5). Therefore, it follows that for every $f \in H^1_A(\mathbb{T})$ one has

$$\|T_m(f)\|_{L^{1,\infty}(\mathbb{T})} \leqslant D'_m \|f\|_{L^1(\mathbb{T})}, \tag{10}$$

where one may take $D'_m = 2D_m$. We shall prove that for every Marcinkiewicz multiplier operator T_m on the torus one has

$$\sup_{\substack{\|f\|_{L^p(\mathbb{T})}\leqslant 1\\ f\in H^p_A(\mathbb{T})}} \|T_m(f)\|_{L^p(\mathbb{T})} \lesssim_{B_m} (p-1)^{-1} \tag{11}$$

as $p \to 1^+$. To do this, we shall make use of the following lemma due to Kislyakov and Xu [8].

Lemma 3 (Kislyakov and Xu, [8] and [11])**.** *If $f \in H^{p_0}_A(\mathbb{T})$ $(0 < p_0 < \infty)$ and $\lambda > 0$, then there exist functions $h_\lambda \in H^\infty_A(\mathbb{T})$, $g_\lambda \in H^{p_0}_A(\mathbb{T})$ and a constant $C_{p_0} > 0$ depending only on p_0 such that*

- $|h_\lambda(x)| \leqslant C_{p_0}\lambda \min\{\lambda^{-1}|f(x)|, |f(x)|^{-1}\lambda\}$ *for all* $x \in \mathbb{T}$,
- $\|g_\lambda\|^{p_0}_{L^{p_0}(\mathbb{T})} \leqslant C_{p_0} \int_{\{x\in\mathbb{T}:|f(x)|>\lambda\}} |f(x)|^{p_0} dx$, *and*
- $f = h_\lambda + g_\lambda$.

We remark that by examining the proof of Lemma 3, one deduces that when $1 \leqslant p_0 \leqslant 2$ the constant C_{p_0} in the statement of the lemma can be chosen independent of p_0. To prove the desired inequality (11) and hence Theorem 1 in the one-dimensional case, we argue as in [11, Theorem 7.4.1]. More precisely, given a $1 < p < 2$, if f is a fixed analytic trigonometric polynomial on $\mathbb{T}$, we first write $\|T_m(f)\|^p_{L^p(\mathbb{T})} = p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{T} : |T_m(f)(x)| > \lambda\}|d\lambda$ and then by using Lemma 3 for $p_0 = 1$ we obtain $\|T_m(f)\|^p_{L^p(\mathbb{T})} \leqslant I_1 + I_2$, where

$$I_1 = p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{T} : |T_m(g_\lambda)(x)| > \lambda/2\}|d\lambda$$

and

$$I_2 = p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{T} : |T_m(h_\lambda)(x)| > \lambda/2\}|d\lambda.$$

To handle I_1, we use the boundedness of T_m from $H^1_A(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$ and Fubini's theorem to deduce that

$$I_1 \lesssim (p-1)^{-1} \int_{\mathbb{T}} |f(x)|^p dx.$$

To obtain appropriate bounds for I_2, we use the boundedness of T_m from $H^2_A(\mathbb{T})$ to $L^2(\mathbb{T})$ and get

$$I_2 \lesssim \int_0^\infty \lambda^{p-3}(\int_{\{x\in\mathbb{T}:|f(x)|\leqslant\lambda\}} |f(x)|^2 dx)d\lambda + \int_0^\infty \lambda^{p+1}(\int_{\{x\in\mathbb{T}:|f(x)|>\lambda\}} |f(x)|^{-2} dx)d\lambda.$$

Hence, by applying Fubini's theorem to each term, we obtain

$$I_2 \lesssim (2-p)^{-1} \int_{\mathbb{T}} |f(x)|^p dx + (p+2)^{-1} \int_{\mathbb{T}} |f(x)|^p dx.$$

Combining the estimates for I_1 and I_2 and using the density of analytic trigonometric polynomials in $(H^p_A(\mathbb{T}), \|\cdot\|_{L^p(\mathbb{T})})$, (11) follows.

To prove the d-dimensional case, take f to be an analytic trigonometric polynomial on $\mathbb{T}^d$ and note that if T_{m_j} are periodic Marcinkiewicz multiplier operators $(j = 1, \cdots, d)$, then for fixed $(x_1, \cdots, x_{d-1}) \in \mathbb{T}^{d-1}$ one can write

$$T_{m_d}(g_{(x_1,\cdots,x_{d-1})})(x_d) = T_{m_1} \otimes \cdots \otimes T_{m_d}(f)(x_1, \cdots, x_d),$$

where

$$g_{(x_1,\cdots,x_{d-1})}(x_d) = T_{m_1} \otimes \cdots \otimes T_{m_{d-1}}(f_{(x_1,\cdots,x_{d-1})})(x_d).$$

Hence, by using (11) in the d-th variable, one deduces that

$$\|T_{m_n}(g_{(x_1,\cdots,x_{d-1})})\|^p_{L^p(\mathbb{T})} \leqslant C^p_{m_d}(p-1)^{-p} \|g_{(x_1,\cdots,x_d)}\|^p_{L^p(\mathbb{T})}$$

where $C_{m_d} > 0$ is the implied constant in (11) corresponding to T_{m_d}. By iterating this argument $d-1$ times, one obtains

$$\|T_{m_1} \otimes \cdots \otimes T_{m_d}(f)\|^p_{L^p(\mathbb{T}^d)} \leqslant [C_{m_1} \cdots C_{m_d}]^p (p-1)^{-dp} \|f\|^p_{L^p(\mathbb{T}^d)}$$

and this completes the proof of Theorem 1.

3.1. **Proof of Corollary 2.** We shall use the multi-dimensional version of Khintchine's inquality: if $(r_k)_{k\in\mathbb{N}_0}$ denotes the set of Rademacher functions indexed by $\mathbb{N}_0$ over a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, then for every finite collection of complex numbers $(a_{k_1,\cdots,k_d})_{k_1,\cdots,k_d\in\mathbb{N}_0}$ one has

$$\Big\| \sum_{k_1,\cdots,k_d\in\mathbb{N}_0} a_{k_1,\cdots,k_d} r_{k_1} \otimes \cdots \otimes r_{k_d} \Big\|_{L^p(\Omega^d)} \sim_p \Big(\sum_{k_1,\cdots,k_d\in\mathbb{N}_0} |a_{k_1,\cdots,k_d}|^2 \Big)^{1/2} \tag{12}$$

for all $0 < p < \infty$. The implied constants do not depend on $(a_{k_1,\cdots,k_d})_{k_1,\cdots,k_d\in\mathbb{N}_0}$, see e.g. Appendix D in [15], and do not blow up as $p \to 1$.

Combining Theorem 1, applied to d-fold tensor products of periodic Marcinkiewicz multiplier operators of the form $\sum_{k\in\mathbb{Z}} \pm\Delta_k$, with the multi-dimensional Khintchine's inequality as in [1, Section 3] shows that the desired bound holds for analytic polynomials. Since analytic trigonometric polynomials on $\mathbb{T}^d$ are dense in $(H^p_A(\mathbb{T}^d), \|\cdot\|_{L^p(\mathbb{T}^d)})$, we deduce that

$$\sup_{\substack{\|f\|_{L^p(\mathbb{T}^d)}\leqslant 1 \\ f\in H^p_A(\mathbb{T}^d)}} \|S_{\mathbb{T}^d}(f)\|_{L^p(\mathbb{T}^d)} \lesssim_d (p-1)^{-d} \quad \text{as} \quad p \to 1^+.$$

It remains to prove the reverse inequality. To do this, for fixed $1 < p \leqslant 2$, choose an $f \in H^p_A(\mathbb{T})$ such that

$$\|S_{\mathbb{T}}(f)\|_{L^p(\mathbb{T})} \geqslant C(p-1)^{-1} \|f\|_{L^p(\mathbb{T})},$$

where $C > 0$ is an absolute constant. The existence of such functions is shown in [12]. Hence, if we define $g \in H^p_A(\mathbb{T}^d)$ by

$$g(x_1, \cdots, x_d) = f(x_1) \cdots f(x_d)$$

for $(x_1, \cdots, x_d) \in \mathbb{T}^d$, then

$$\begin{aligned}\|S_{\mathbb{T}^d}(g)\|_{L^p(\mathbb{T}^d)} &= \|S_{\mathbb{T}}(f)\|_{L^p(\mathbb{T})} \cdots \|S_{\mathbb{T}}(f)\|_{L^p(\mathbb{T})} \\ &\geqslant C^d(p-1)^{-d} \|f\|^d_{L^p(\mathbb{T})} = C^d(p-1)^{-d} \|g\|_{L^p(\mathbb{T}^d)}\end{aligned}$$

and this proves the sharpness of Corollary 2.

Remark 4. For the subspace $H^p_{A,\mathrm{diag}}(\mathbb{T}^d)$ of $L^p(\mathbb{T}^d)$ consisting of functions of the form $f(x_1,\dots,x_d)=F(x_1+\cdots+x_d)$ for some one-variable function $F\in H^p_A(\mathbb{T})$, we have the improved estimate

$$\sup_{\substack{\|f\|_{L^p(\mathbb{T}^d)}\leqslant 1\\ f\in H^p_{A,\mathrm{diag}}(\mathbb{T}^d)}} \|S_{\mathbb{T}^d}(f)\|_p \sim (p-1)^{-1}, \quad p\to 1^+.$$

This follows from invariance of the L^p-norm and Fubini's theorem which allow us to reduce to the one-dimensional case. On the other hand, the natural inclusion of $H^p_A(\mathbb{T}^k)$ in $H^p_A(\mathbb{T}^d)$ yields examples of subspaces with sharp blowup of order $(p-1)^{-k}$ for any $k=1,\dots,d-1$.

Both the original proof of Pichorides's theorem and the extension in this paper rely on complex-analytic techniques, via canonical factorisation in [12] and conjugate functions in [11]. However, a complex-analytic structure is not necessary in order for an estimate of the form in Corollary 2 to hold. For instance, the same conclusion remains valid for $g\in \tilde{H}^p_0(\mathbb{T}^d)$, the subset of $L^p(\mathbb{T}^d)$ consisting of functions with $\mathrm{supp}(\hat{f})\subset(-\mathbb{N})^d$. Moreover, for functions of the form $f+g$, where $f\in H^p_A(\mathbb{T}^d)$ and $g\in\tilde{H}^p_0(\mathbb{T}^d)$ with $\|f\|_p=\|g\|_p\leqslant 1/2$, we then have $\|f+g\|_p\leqslant 1$ and $\|S_{\mathbb{T}^n}(f+g)\|_p\leqslant\|S_{\mathbb{T}^d}(f)\|_p+\|S_{\mathbb{T}^d}(g)\|_p\lesssim(p-1)^{-d}$ as $p\to 1^+$.

3.2. **A transference theorem.** In this subsection, we explain how one can transfer the aforementioned result of Tao and Wright on boundedness of Marcinkiewicz multiplier operators from $H^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$ to the periodic setting. To this end, let us first recall the definition of the local Hardy space $h^1(\mathbb{R})$ introduced by D. Goldberg [6], which can be described as the space of L^1-functions for which the "high-frequency"part belongs to $H^1(\mathbb{R})$. Namely, if we take ϕ to be a smooth function supported in $[-1,1]$ and such that $\phi|_{[-1/2,1/2]}\equiv 1$, and set $\psi=1-\phi$, one has that $f\in h^1(\mathbb{R})$ if, and only if,

$$\|f\|_{h^1(\mathbb{R})}=\|f\|_{L^1(\mathbb{R})}+\|T_\psi(f)\|_{H^1(\mathbb{R})}<+\infty.$$

The desired transference result is a consequence of D. Chen's [3, Thm. 29].

Theorem 5. *If λ is a continuous and bounded function on $\mathbb{R}$ such that*

$$\|T_\lambda(f)\|_{L^{1,\infty}(\mathbb{R})}\leqslant C_1\|f\|_{h^1(\mathbb{R})},$$

then

$$\|T_{\lambda|_{\mathbb{Z}}}(g)\|_{L^{1,\infty}(\mathbb{T})}\leqslant C_2\|g\|_{H^1(\mathbb{T})}.$$

Observe that given a Marcinkiewicz multiplier m on the torus, one can construct a Marcinkiewicz multiplier λ on $\mathbb{R}$ such that $\lambda|_{\mathbb{Z}}=m$. Indeed, it suffices to take λ to be continuous such that $\lambda(n)=m(n)$ for every $n\in\mathbb{Z}$ and affine on the intervals of the form $(n,n+1)$, $n\in\mathbb{Z}$.

In order to use Theorem 5, let ψ be as above and consider the "high-frequency" part λ_+ of λ given by $\lambda_+=\psi\lambda$. Note that for every Schwartz function f we may write

$$T_{\lambda_+}(f)=T_\lambda(\widetilde{f}),$$

where $\widetilde{f}=T_\psi(f)$. We thus deduce that

$$\|T_{\lambda_+}(f)\|_{L^{1,\infty}(\mathbb{R})}=\|T_\lambda(\widetilde{f})\|_{L^{1,\infty}(\mathbb{R})}\lesssim\|\widetilde{f}\|_{H^1(\mathbb{R})}\lesssim\|f\|_{h^1(\mathbb{R})},$$

and hence, Theorem 5 yields that $T_{\lambda_+|_{\mathbb{Z}}}$ is bounded from $H^1(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$. Since for every trigonometric polynomial g we can write

$$T_m(g)=T_0(g)+T_{\lambda_+|_{\mathbb{Z}}}(g),$$

where $T_0(g)=m(0)\widehat{g}(0)$, and we have that

$$\|T_0(g)\|_{L^{1,\infty}(\mathbb{T})}=|m(0)||\widehat{g}(0)|\leqslant\|m\|_{\ell^\infty(\mathbb{Z})}\|g\|_{L^1(\mathbb{T})}\leqslant\|m\|_{\ell^\infty(\mathbb{Z})}\|g\|_{H^1(\mathbb{T})},$$

it follows that $\|T_m(g)\|_{L^{1,\infty}(\mathbb{T})} \lesssim \|g\|_{H^1(\mathbb{T})}$.

4. Another application of Theorem 1

In [19], Zygmund showed that there exists a constant $C > 0$ such that for every $f \in H^1_A(\mathbb{T})$, we have

$$\|S_{\mathbb{T}}(f)\|_{L^1(\mathbb{T})} \leqslant C\|f\|_{L\log L(\mathbb{T})}. \tag{13}$$

Note that if one removes the assumption that $f \in H^1_A(\mathbb{T})$, then the Orlicz space $L\log L(\mathbb{T})$ must be replaced by the smaller space $L\log^{3/2} L(\mathbb{T})$, see [1].

Zygmund's proof again relies on canonical factorisation in $H^p_A(\mathbb{T})$, but a higher-dimensional extension of (13) can now be obtained from Theorem 1. Towards this aim, we use the following variant of S. Yano's classical extrapolation theorem [18] for multiplier operators acting on Hardy spaces, which might be of independent interest.

Lemma 6. *Let T_m be a multiplier operator acting on functions defined over $\mathbb{T}^d$. Assume that for some $p_0 > 1$ one has*

$$\sup_{\substack{\|f\|_{L^p(\mathbb{T}^d)}\leqslant 1 \\ f\in H^p_A(\mathbb{T}^d)}} \|T_m(f)\| \leqslant C(p-1)^{-r} \tag{14}$$

for all $1 < p \leqslant p_0$, where $C > 0$, $r > 0$ are absolute constants. Then, there exists a constant $C_r > 0$ such that

$$\|T_m(g)\|_{L^1(\mathbb{T}^d)} \leqslant C_r\|g\|_{L\log^r L(\mathbb{T}^d)} \tag{15}$$

for every analytic trigonometric polynomial g on $\mathbb{T}^d$.

Proof. Consider a trigonometric polynomial f on $\mathbb{T}^d$ and note that if P_j denotes the Riesz projection in the j-th variable, $j = 1, \cdots, d$, then $(P_1 \otimes \cdots \otimes P_d)(f)$ is an analytic trigonometric polynomial on $\mathbb{T}^d$. Hence, by using duality of analytic Hardy spaces, Parseval's identity twice, and (11), one can easily see that

$$\|T_{\overline{m}} \circ (P_1 \otimes \cdots \otimes P_d)(f)\|_{L^q(\mathbb{T}^d)} \leqslant Cq^d\|f\|_{L^q(\mathbb{T}^d)}$$

for every $2 \leqslant q < \infty$. Therefore, by using the inequality above and the density of trigonometric polynomials in $(C(\mathbb{T}^d), \|\cdot\|_{L^\infty(\mathbb{T}^d)})$, we deduce that

$$\|T_{\overline{m}} \circ (P_1 \otimes \cdots \otimes P_d)(h)\|_{\exp L^{1/d}(\mathbb{T}^d)} \leqslant C\|h\|_{L^\infty(\mathbb{T}^d)} \tag{16}$$

for every continuous function h on $\mathbb{T}^d$.

Take now an analytic trigonometric polynomial g on $\mathbb{T}^d$ and note that $T_m(g)$ is an element of $L^1(\mathbb{T}^d)$. Hence, if μ is a measure on $\mathbb{T}^d$ given by $d\mu(x_1, \cdots, x_d) = T_m(g)(x_1, \cdots, x_d)dx_1 \cdots dx_d$, then its total variation satisfies $\|\mu\| = \|T_m(g)\|_{L^1(\mathbb{T}^d)}$ and thus, by using duality, Parseval's identity twice, as well as (6) and (16), we have

$$\begin{aligned}\|T_m(g)\|_{L^1(\mathbb{T}^d)} &= \sup_{\substack{\|h\|_{L^\infty(\mathbb{T}^d)}\leqslant 1\\ h\in C(\mathbb{T}^d)}} \Big|\int_{\mathbb{T}^d} T_m(g)(x)\overline{h(x)}dx\Big| \\ &= \sup_{\substack{\|h\|_{L^\infty(\mathbb{T}^d)}\leqslant 1\\ h\in C(\mathbb{T}^d)}} \Big|\sum_{k_1,\cdots,k_d\in\mathbb{N}_0} m(k_1,\cdots,k_d)\widehat{g}(k_1,\cdots,k_d)\overline{\widehat{h}(k_1,\cdots,k_d)}\Big| \\ &= \sup_{\substack{\|h\|_{L^\infty(\mathbb{T}^d)}\leqslant 1\\ h\in C(\mathbb{T}^d)}} \Big|\int_{\mathbb{T}^d} T_{\overline{m}}\circ(P_1\otimes\cdots\otimes P_d)(h)(x)\overline{g(x)}dx\Big| \\ &\lesssim_r \|g\|_{L\log^r L(\mathbb{T}^d)} \sup_{\substack{\|h\|_{L^\infty(\mathbb{T}^d)}\leqslant 1\\ h\in C(\mathbb{T}^d)}} \|T_{\overline{m}}\circ(P_1\otimes\cdots\otimes P_d)(h)\|_{\exp L^{1/r}(\mathbb{T}^d)} \\ &\lesssim \|g\|_{L\log^r L(\mathbb{T}^d)}.\end{aligned}$$

□

Proposition 7. *Given $d\in\mathbb{N}$, there exists a constant $C_d>0$ such that for every analytic trigonometric polynomial g on $\mathbb{T}^d$ one has*

$$\|S_{\mathbb{T}^d}(g)\|_{L^1(\mathbb{T}^d)} \leqslant C_d\|g\|_{L\log^d L(\mathbb{T}^d)}. \tag{17}$$

The exponent $r=d$ in the Orlicz space $L\log^d L(\mathbb{T}^d)$ cannot be improved.

Proof. Let $T_{\omega_j}=\sum_{k\in\mathbb{Z}} r_k(\omega_j)\Delta_j$ denote a randomised version of $S_{\mathbb{T}}$, $j=1,\cdots,d$. By Theorem 1, one deduces that $T_{\omega_1}\otimes\cdots\otimes T_{\omega_d}$ satisfies the assumptions of Lemma 6 with $r=d$. Hence, we get

$$\|(T_{\omega_1}\otimes\cdots\otimes T_{\omega_d})(g)\|_{L^1(\mathbb{T}^d)} \leqslant A_d\|g\|_{L\log^d L(\mathbb{T}^d)} \tag{18}$$

and so, the proof of (17) is obtained by using the last inequality and (12).

To prove sharpness for $d=1$, let N be a large positive integer to be chosen later and take $V_{2^N}=2K_{2^{N+1}}-K_{2^N}$ to be the de la Vallée Poussin kernel of order 2^N, where K_n denotes the Fejér kernel of order n, $K_n(x)=\sum_{|j|\leqslant n}[1-|j|/(n+1)]e^{i2\pi jx}$. Consider the function f_N by

$$f_N(x)=e^{i2\pi 2^{N+1}x}V_{2^N}(x).$$

Then, one can easily check that $f_N\in H^1_A(\mathbb{T})$, $\Delta_{N+1}(f_N)(x)=\sum_{k=2^N}^{2^{N+1}-1} e^{i2\pi kx}$ and $\|f_N\|_{L\log^r L(\mathbb{T})}\lesssim N^r$. Hence, if we assume that (13) holds for some $L\log^r L(\mathbb{T})$, then we see that we must have

$$N\lesssim \|\Delta_{N+1}(f_N)\|_{L^1(\mathbb{T})} \leqslant \|S_{\mathbb{T}}(f_N)\|_{L^1(\mathbb{T})} \lesssim \|f_N\|_{L\log^r L(\mathbb{T})} \lesssim N^r$$

and so, if N is large enough, it follows that $r\geqslant 1$, as desired. To prove sharpness in the d-dimensional case, take $g_N(x_1,\cdots,x_d)=f_N(x_1)\cdots f_N(x_d)$, f_N being as above, and note that

$$N^d\lesssim \|\Delta_{N+1}(f_N)\|^d_{L^1(\mathbb{T})} \leqslant \|S_{\mathbb{T}^d}(g_N)\|_{L^1(\mathbb{T}^d)} \lesssim \|g_N\|_{L\log^r L(\mathbb{T}^d)} \lesssim N^r.$$

Hence, by taking $N\to\infty$, we deduce that $r\geqslant d$. □

Remark 8. We remark that one can prove Proposition 7 directly by using the method of Section 3 and in particular without appealing to Lemma 6. More specifically, by using Lemma 3 and an interpolation argument analogous to the one presented in Section 3, one can easily show that if T is a sublinear operator that is bounded from $H^1_A(\mathbb{T})$ to $L^{1,\infty}(\mathbb{T})$ and bounded from $H^2_A(\mathbb{T})$ to $L^2(\mathbb{T})$, then for every $r \geqslant 0$ one has

$$\int_{\mathbb{T}} |T(f)(x)| \log^r(1 + |T(f)(x)|)dx \leqslant C_r[1 + \int_{\mathbb{T}} |f(x)| \log^{r+1}(1 + |f(x)|)dx] \quad (19)$$

for every analytic trigonometric polynomial f on $\mathbb{T}$, where $C_r > 0$ is a constant depending only on r. Hence, by using (19), iteration as well as the definition of $\| \cdot \|_{L \log^d L(\mathbb{T}^d)}$, (18) follows.

Furthermore, note that by using the aforementioned interpolation argument one can actually show that there exists a constant $A_d > 0$, depending only on d, such that

$$\|S_{\mathbb{T}^d}(f)\|_{L^{1,\infty}(\mathbb{T}^d)} \leqslant A_d \|f\|_{L \log^{d-1} L(\mathbb{T}^d)} \quad (20)$$

for every analytic trigonometric polynomial f on $\mathbb{T}^d$. Notice that if we remove the assumption that f is analytic, then the Orlicz space $L \log^{d-1} L(\mathbb{T}^d)$ in (20) must be replaced by $L \log^{3d/2-1} L(\mathbb{T}^d)$, see [1].

5. Euclidean variants of Theorem 1

In this section we obtain an extension of Pichorides's theorem to the Euclidean setting. Our result will be a consequence of the following variant of Marcinkiewicz-type interpolation on Hardy spaces.

Proposition 9. *Assume that T is a sublinear operator that satisfies:*

- $\|T(f)\|_{L^{1,\infty}(\mathbb{R})} \leqslant C\|f\|_{L^1(\mathbb{R})}$ *for all* $f \in H^1_A(\mathbb{R})$ *and*
- $\|T(f)\|_{L^2(\mathbb{R})} \leqslant C\|f\|_{L^2(\mathbb{R})}$ *for all* $f \in H^2_A(\mathbb{R})$,

where $C > 0$ is an absolute constant. Then, for every $1 < p < 2$, T maps $H^p_A(\mathbb{R})$ to $L^p(\mathbb{R})$ and moreover,

$$\|T\|_{H^p_A(\mathbb{R}) \to L^p(\mathbb{R})} \lesssim [(p-1)^{-1} + (2-p)^{-1}]^{1/p}.$$

Proof. Fix $1 < p < 2$ and take an $f \in H^p_A(\mathbb{R})$. From a classical result due to Peter Jones [7, Theorem 2] it follows that for every $\lambda > 0$ one can write $f = F_\lambda + f_\lambda$, where $F_\lambda \in H^1_A(\mathbb{R})$, $f_\lambda \in H^\infty_A(\mathbb{R})$ and, moreover, there is an absolute constant $C_0 > 0$ such that

- $\int_{\mathbb{R}} |F_\lambda(x)|dx \leqslant C_0 \int_{\{x \in \mathbb{R}: N(f)(x) > \lambda\}} N(f)(x)dx$ and
- $\|f_\lambda\|_{L^\infty(\mathbb{R})} \leqslant C_0\lambda$.

Here, $N(f)$ denotes the non-tangential maximal function of $f \in H^p_A(\mathbb{R})$ given by

$$N(f)(x) = \sup_{|x-x'|<t} |(f * P_t)(x')|,$$

where, for $t > 0$, $P_t(s) = t/(s^2 + t^2)$ denotes the Poisson kernel on the real line. Hence, by using the Peter Jones decomposition of f, we have

$$\|T(f)\|^p_{L^p(\mathbb{R})} = \int_0^\infty p\lambda^{p-1}|\{x \in \mathbb{R} : |T(f)(x)| > \lambda/2\}|d\lambda \leqslant I_1 + I_2,$$

where

$$I_1 = p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{R} : |T(F_\lambda)(x)| > \lambda/2\}|d\lambda$$

and

$$I_2 = p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{R} : |T(f_\lambda)(x)| > \lambda/2\}|d\lambda.$$

We shall treat I_1 and I_2 separately. To bound I_1, using our assumption on the boundedness of T from $H^1_A(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$ together with Fubini's theorem, we deduce that there is an absolute constant $C_1 > 0$ such that

$$I_1 \leqslant C_1(p-1)^{-1}\int_{\mathbb{R}}[N(f)(x)]^p dx. \tag{21}$$

To bound the second term, we first use the boundedness of T from $H^2_A(\mathbb{R})$ to $L^2(\mathbb{R})$ as follows

$$I_2 \leqslant C\int_0^\infty p\lambda^{p-3}\Big(\int_{\mathbb{R}}|f_\lambda(x)|^2dx\Big)d\lambda$$

and then we further decompose the right-hand side of the last inequality as $I_{2,\alpha} + I_{2,\beta}$, where

$$I_{2,\alpha} = C\int_0^\infty p\lambda^{p-3}\Big(\int_{\{x\in\mathbb{R}:N(f)(x)>\lambda\}}|f_\lambda(x)|^2dx\Big)d\lambda$$

and

$$I_{2,\beta} = C\int_0^\infty p\lambda^{p-3}\Big(\int_{\{x\in\mathbb{R}:N(f)(x)\leqslant\lambda\}}|f_\lambda(x)|^2dx\Big)d\lambda.$$

The first term $I_{2,\alpha}$ can easily be dealt with by using the fact that $\|f_\lambda\|_{L^\infty(\mathbb{R})} \leqslant C_0\lambda$,

$$I_{2,\alpha} \leqslant C'\int_0^\infty p\lambda^{p-1}|\{x\in\mathbb{R}: N(f)(x)>\lambda\}|d\lambda = C'\int_{\mathbb{R}}[N(f)(x)]^pdx,$$

where $C' = C_0C$. To obtain appropriate bounds for $I_{2,\beta}$, note that since $|f_\lambda|^2 = |f-F_\lambda|^2 \leqslant 2|f|^2 + 2|F_\lambda|^2$, one has $I_{2,\beta} \leqslant I'_{2,\beta} + I''_{2,\beta}$, where

$$I'_{2,\beta} = 2C\int_0^\infty p\lambda^{p-3}\Big(\int_{\{x\in\mathbb{R}:N(f)(x)\leqslant\lambda\}}|f(x)|^2dx\Big)d\lambda$$

and

$$I''_{2,\beta} = 2C\int_0^\infty p\lambda^{p-3}\Big(\int_{\{x\in\mathbb{R}:N(f)(x)\leqslant\lambda\}}|F_\lambda(x)|^2dx\Big)d\lambda.$$

To handle $I'_{2,\beta}$, note that since $f \in H^p_A(\mathbb{R})$ $(1<p<2)$ one has $|f(x)| \leqslant N(f)(x)$ for a.e. $x \in \mathbb{R}$ and hence, by using this fact together with Fubini's theorem, one obtains

$$I'_{2,\beta} \leqslant C(2-p)^{-1}\int_{\mathbb{R}}[N(f)(x)]^pdx.$$

Finally, for the last term $I''_{2,\beta}$, we note that for a.e. x in $\{N(f)\leqslant\lambda\}$ one has

$$|F_\lambda(x)| \leqslant |f(x)| + |f_\lambda(x)| \leqslant N(f)(x) + |f_\lambda(x)| \leqslant (1+C_0)\lambda$$

and hence,

$$\begin{aligned} I''_{2,\beta} &\leqslant C''\int_0^\infty \lambda^{p-2}\Big(\int_{\mathbb{R}}|F_\lambda(x)|dx\Big)d\lambda \\ &\leqslant C''\int_0^\infty \lambda^{p-2}\Big(\int_{\{x\in\mathbb{R}:N(f)(x)>\lambda\}}N(f)(x)dx\Big)d\lambda \leqslant C''(p-1)^{-1}\int_{\mathbb{R}}[N(f)(x)]^pdx, \end{aligned}$$

where $C'' = 4(1+C_0)$ and in the last step we used Fubini's theorem. Since $I_2 \leqslant I_{2,\alpha} + I'_{2,\beta} + I''_{2,\beta}$, we conclude that there is a $C_2 > 0$ such that

$$I_2 \leqslant C_2[(p-1)^{-1} + (2-p)^{-1}]\int_{\mathbb{R}}[N(f)(x)]^pdx. \tag{22}$$

It thus follows from (21) and (22) that

$$\|T(f)\|_{L^p(\mathbb{R})} \lesssim [(p-1)^{-1} + (2-p)^{-1}]^{1/p}\|N(f)\|_{L^p(\mathbb{R})}.$$

To complete the proof of the proposition note that one has

$$\|N(f)\|_{L^p(\mathbb{R})} \leqslant C_p\|f\|_{L^p(\mathbb{R})} \ (f \in H^p_A(\mathbb{R})), \tag{23}$$

where one can take $C_p = A_0^{1/p}$, $A_0 \geqslant 1$ being an absolute constant, see e.g. p.278-279 in vol.I in [20], where the periodic case is presented. The Euclidean version is completely analogous. Hence, if $1 < p < 2$, one deduces that the constant C_p in (23) satisfies $C_p \leqslant A_0$ and so, we get the desired result. □

Using the above proposition and iteration, we obtain the following Euclidean version of Theorem 1.

Theorem 10. *Let $d \in \mathbb{N}$ be a given dimension. If T_{m_j} is a Marcinkiewicz multiplier operator on $\mathbb{R}$ $(j = 1, \cdots, d)$, then*

$$\|T_{m_1} \otimes \cdots \otimes T_{m_d}\|_{H^p_A(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \lesssim_{C_{m_1}, \cdots, C_{m_d}} (p-1)^{-d}$$

as $p \to 1^+$, where $C_{m_j} = \|m_j\|_{L^\infty(\mathbb{R})} + A_{m_j}$, A_{m_j} is as in (7), $j = 1, \cdots, d$.

A variant of Pichorides's theorem on $\mathbb{R}^d$ now follows from Theorem 10 and (12). To formulate our result, for $k \in \mathbb{Z}$, define the rough Littlewood-Paley projection P_k to be a multiplier operator given by

$$\widehat{P_k(f)} = [\chi_{[2^k, 2^{k+1})} + \chi_{(-2^{k+1}, -2^k]}]\widehat{f}.$$

For $d \in \mathbb{N}$, define the d-parameter rough Littlewood-Paley square function $S_{\mathbb{R}^d}$ on $\mathbb{R}^d$ by

$$S_{\mathbb{R}^d}(f) = \Big(\sum_{k_1, \cdots, k_d \in \mathbb{Z}} |P_{k_1} \otimes \cdots \otimes P_{k_d}(f)|^2 \Big)^{1/2}$$

for f initially belonging to the class of Schwartz functions on $\mathbb{R}^d$. Arguing as in Subsection 3.1, we get a Euclidean version of Corollary 2 as a consequence of Theorem 10.

Corollary 11. *For $d \in \mathbb{N}$, one has*

$$\|S_{\mathbb{R}^d}\|_{H^p_A(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \sim_d (p-1)^{-d}$$

as $p \to 1^+$.

Remark 12. The multiplier operators covered in Theorem 10 are properly contained in the class of general multi-parameter Marcinkiewicz multiplier operators treated in Theorem 6′ in Chapter IV of [15]. For a class of smooth multi-parameter Marcinkiewicz multipliers M. Wojciechowski [17] proves that their $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ operator norm is of order $(p-1)^{-d}$ and that they are bounded on the d-parameter Hardy space $H^p(\mathbb{R} \times \cdots \times \mathbb{R})$ for all $1 \leqslant p \leqslant 2$. Note that the multi-parameter Littlewood-Paley square function is not covered by this result; see also [1] for more refined negative statements.

References

[1] Bakas, Odysseas. *Endpoint Mapping properties of the Littlewood-Paley square function.* preprint arXiv:1612.09573 (2016).

[2] Bourgain, Jean. *On the behavior of the constant in the Littlewood-Paley inequality.* In: Geometric Aspects of Functional Analysis (1987-88), pp. 202-208. Lecture notes in math. 1376, Springer Berlin, 1989.

[3] Chen, Danin. *Multipliers on certain function spaces.* PhD thesis, University of Winsconsin-Milwaukee, 1998.

[4] Duren, Peter L. *Theory of H^p spaces.* Vol. 38. New York: Academic press, 1970.

[5] Edwards, Robert E., and Garth Ian Gaudry. *Littlewood-Paley and multiplier theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 90. Springer-Verlag Berlin-New York, 1977.

[6] Goldberg, David. *A local version of real Hardy spaces.* PhD thesis, Princeton University, 1978.

[7] Jones, Peter Wilcox. *L^∞ estimates for the $\bar{\partial}$ problem in a half-plane.* Acta Math. 150, no.1-2 (1983): 137–152.
[8] Kislyakov, Serguei Vital'evich, and Quanhua Xu. *Real interpolation and singular integrals.* Algebra i Analiz 8, no. 4 (1996): 75-109.
[9] Krasnosel'skii, Mark Aleksandrovich, and Iakov Bronislavovich Rutitskii. *Convex functions and Orlicz spaces.* P. Noordhoff, Groningen, 1961.
[10] Marcinkiewicz, Józef. *Sur les multiplicateurs des séries de Fourier.* Studia Math. 8 (1939): 78-91.
[11] Pavlović, Miroslav. *Introduction to function spaces on the disk.* Posebna Izdanja [Special editions] 20, Matematicki Institut SANU, Belgrade, 2004.
[12] Pichorides, Stylianos K. *A remark on the constants of the Littlewood-Paley inequality.* Proc. Amer. Math. Soc. 114, no. 3 (1992): 787-789.
[13] Rubel, Lee A. and Allen L. Shields. *The failure of interior-exterior factorization in the polydisc and the ball.* Tohoku Math. J. 24 (1972): 409-413.
[14] Rudin, Walter. *Function theory in polydiscs.* W.A. Benjamin, New York and Amsterdam, 1969.
[15] Stein, Elias M. *Singular integrals and differentiability properties of functions* (PMS-30). Vol. 30. Princeton university press, 2016.
[16] Tao, Terence, and James Wright. *Endpoint multiplier theorems of Marcinkiewicz type.* Rev. Mat. Iberoam. 17, no. 3 (2001): 521-558.
[17] Wojciechowski, Michał. *A Marcinkiewicz type multiplier theorem for H^1 spaces on product domains.* Studia Math. 140, no. 3 (2000): 273-287.
[18] Yano, Shigeki. *Notes on Fourier Analysis (XXIX).* An extrapolation theorem. J. Math. Soc. of Japan 3, no. 2 (1951): 296-305.
[19] Zygmund, Antoni. *On the convergence and summability of power series on the circle of convergence (I).* Fundamenta Math. 30 (1938): 170-196.
[20] Zygmund, Antoni. *Trigonometric series.* Vol. I, II. Cambridge University Press, 2002.

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