REARRANGEMENT ESTIMATES FOR A_∞ WEIGHTS

IRINA ASEKRITOVA, MARIA J. CARRO, NATAN KRUGLYAK, AND JAVIER SORIA

Abstract. We find a new characterizations of an A_∞ weight ω, in terms of the decreasing rearrangement of the restriction of ω to cubes Q.

1. Introduction

In order to make our exposition complete, let us start by reminding the definition and some properties of the class of Muckenhoupt weights A_p [10, 5]. Everywhere below, Q denotes a cube in $\mathbb{R}^n$ with edges parallel to the coordinate axes. A positive weight w on $\mathbb{R}^n$ belongs to A_p, for $1 < p < \infty$, if

$$\|\omega\|_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,dx\right)^{p-1} < \infty,$$

where, as usual, $\frac{1}{p}+\frac{1}{p'}=1$. It is immediate to prove that $\omega \in A_p$ if and only if $\omega^{1-p'} \in A_{p'}$ and that $A_q \subset A_p$ if $q < p$. Let us also remind that, by definition,

$$A_\infty = \cup_{p>1} A_p.$$

In fact, there are many different (and obviously equivalent) definitions of A_∞ weights (see [7, 3] and the references therein). In particular, we explicitly refer to Definition 2.5 in [3], where 12 different characterizations are given for the standard bases of cubes, the one we are going to consider in this work.

The first main goal of this paper is to present some new characterizations of this class of weights in the setting of rearrangement inequalities and show that some of the previous equivalences can be simplified using simple techniques.

For our purposes, let us now recall that, given a positive σ-finite measure μ on $\mathbb{R}^n$, $f^{*,\mu}$ is the decreasing rearrangement of f with respect to the measure μ:

$$f^{*,\mu}(t) = \inf\{s > 0 : \mu(\{x : |f(x)| > s\}) \le t\}.$$

All over the paper, μ will be of the form $d\mu(x) = \omega(x)dx$, where ω is a weight; that is, a nonnegative and locally integrable function in $\mathbb{R}^n$.

2010 *Mathematics Subject Classification.* 42B25; 46E30.

Key words and phrases. Muckenhoupt weights; reverse Hölder inequality; decreasing rearrangement.

The second and fourth authors were partially supported by Grants MTM2016-75196-P (MINECO/FEDER, UE) and 2017SGR358.

Given a cube Q, we shall write ω_Q the restriction of ω on Q; i.e., $\omega_Q = \omega\chi_Q$. We shall also use w_Q^{-1} to indicate $w^{-1}\chi_Q$. As usual,

$$\omega(Q) = \int_Q \omega(x)dx,$$

and we say that ω is a doubling weight if there exists $d > 0$ so that, for all cubes Q

$$\omega(2Q) \leq d\omega(Q). \tag{1}$$

Also, given a cube Q, we shall work with dyadic families of cubes $\{Q_i\}_i$ with respect to Q, which simply means a family of cubes which are obtained by subdivision of Q in standard dyadic way.

Finally, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Main Result

In Theorem 2.10, we are going to describe a new characterization of A_∞ weights w, which is determined by a simple condition on the behavior of the decreasing rearrangement of w compared to a suitable power function (see Definition 2.3). For this purpose, we start with some preliminary results.

Lemma 2.1. *Let $f > 0$ be a nonincreasing function on the interval $[0, R)$, $R > 0$. Then the following two conditions are equivalent:*

(i) *there exists a constant $\gamma \geq 1$ such that, for all $t \in [0, R)$,*

$$\frac{1}{t}\int_0^t f(\tau)\,d\tau \leq \gamma f(t); \tag{2}$$

(ii) *there exist constants $\gamma \geq 1$, $0 < \lambda < 1$ such that, for all $0 < s \leq t < R$, we have*

$$\frac{f(s)}{f(t)} \leq \gamma\Big(\frac{s}{t}\Big)^{-\lambda}. \tag{3}$$

Moreover, under one (and then, both) of these conditions, there exist constants $\gamma \geq 1$ and $0 < \lambda < 1$ such that for all $0 < s \leq t < R$,

$$\int_0^s f(\tau)\,d\tau \leq \gamma\Big(\frac{s}{t}\Big)^{1-\lambda}\int_0^t f(\tau)\,d\tau. \tag{4}$$

Proof. We show first that (i) $\Rightarrow$ (ii): Let $0 < s \leq t < R$ and let $\delta > 0$. Then

$$\frac{s}{t}f(s)^{1+\delta} \leq \frac{2s}{st}\int_{s/2}^s f(\tau)^{1+\delta}\,d\tau \leq \frac{2}{t}\int_{s/2}^t f(\tau)^{1+\delta}\,d\tau. \tag{5}$$

Since

$$\frac{1}{t}\int_{s/2}^{t} f(\tau)^{1+\delta}\,d\tau = \frac{1}{t}\int_{s/2}^{t} f(\tau)f(\tau)^{\delta}\,d\tau \le \frac{1}{t}\int_{s/2}^{t}\Big(\int_{s/2}^{\tau} f(u)\,du\Big)'\Big(\frac{1}{\tau}\int_{0}^{\tau} f(u)\,du\Big)^{\delta}\,d\tau,$$

we have, using integration by parts (disregarding terms, which are negative) and estimate (2), that

$$\begin{aligned}\frac{1}{t}\int_{s/2}^{t} f(\tau)^{1+\delta}\,d\tau &\le \Big(\frac{1}{t}\int_{s/2}^{t} f(u)\,du\Big)\Big(\frac{1}{t}\int_{0}^{t} f(u)\,du\Big)^{\delta} \\ &\quad + \frac{1}{t}\int_{s/2}^{t}\frac{\delta}{\tau^{1+\delta}}\Big(\int_{0}^{\tau} f(u)\,du\Big)^{\delta+1}\,d\tau \\ &\le \Big(\frac{1}{t}\int_{0}^{t} f(u)\,du\Big)^{\delta+1} + \frac{\delta\gamma^{1+\delta}}{t}\int_{s/2}^{t} f(\tau)^{1+\delta}\,d\tau.\end{aligned}$$

Therefore,

$$(1-\delta\gamma^{1+\delta})\frac{1}{t}\int_{s/2}^{t} f(\tau)^{1+\delta}\,d\tau \le \Big(\frac{1}{t}\int_{0}^{t} f(u)\,du\Big)^{1+\delta} \le \gamma^{1+\delta}f(t)^{1+\delta}.$$

So if choose $\delta < \gamma^{-(1+\delta)}$, we have by (5), that

$$\frac{s}{t}f(s)^{1+\delta} \le \frac{2\gamma^{1+\delta}}{1-\delta\gamma^{1+\delta}}f(t)^{1+\delta},$$

and hence, taking $\lambda = \frac{1}{1+\delta}$, we obtain

$$\frac{f(s)}{f(t)} \le \tilde{\gamma}\Big(\frac{s}{t}\Big)^{-\lambda},$$

with $\tilde{\gamma} = 2^{\frac{1}{1+\delta}}\gamma(1-\delta\gamma^{1+\delta})^{-\frac{1}{1+\delta}}$.

Conversely, (ii) $\Rightarrow$ (i) follows from (3) by integration

$$\int_0^t f(\tau)\,d\tau \le \gamma f(t)t^{\frac{1}{1+\delta}}\int_0^t \tau^{-\frac{1}{1+\delta}}\,d\tau = \gamma\frac{1+\delta}{\delta}tf(t).$$

Finally, (4) is an immediate consequence of (2) and (3). □

The following lemma is an easy modification of [2, Lemma 7.2] and it will be used both for μ the Lebesgue measure and $d\mu(x) = \omega(x)dx$.

Lemma 2.2. *Let us assume that μ is an absolutely continuous doubling measure. Let $0 < \alpha < 1$ and let $\Omega \subset Q$ so that $0 < \mu(\Omega) < \alpha\mu(Q)$. Then, there exists a family of dyadic cubes with respect to Q, $Q_i \subset Q$, $i \in I$ so that:*

(i) $\mathring{Q}_i \cap \mathring{Q}_j = \emptyset$, $\;i \neq j, i, j \in I$;
(ii) $\mu(\Omega \setminus \cup_{i\in I} Q_i) = 0$;

(iii) *for all cubes* $Q_i, i \in I$, *we have*

$$\frac{\alpha}{d^2}\mu(Q_i) \leq \mu(\Omega \cap Q_i) < \alpha\mu(Q_i),$$

where d *is the doubling constant in* (1).

Proof. The family of cubes $\{Q_i\}_i$ are the maximal dyadic subcubes of Q so that

$$\mu(\Omega \cap Q_i) < \alpha\mu(Q_i)$$

and at least one of its children $\tilde{Q}_i$ satisfies the opposite inequality

$$\mu(\Omega \cap \tilde{Q}_i) \geq \alpha\mu(\tilde{Q}_i).$$

It is clear that the set of such cubes is not empty, since otherwise we would have that, for all dyadic subcubes Q_i, $\mu(\Omega \cap Q_i) < \alpha\mu(Q_i)$. Thus, using the Lebesgue differentiation theorem for μ (which is true for doubling measures) we would obtaing that $\chi_\Omega(x) = 0$, a.e. $x \in \Omega$, which is a contradiction. Now, by construction, property (i) and the right hand-side of (iii) trivially hold. Also, if $\mu(\Omega \cap \tilde{Q}_i) \geq \alpha\mu(\tilde{Q}_i)$, then $Q_i \subset 4\tilde{Q}_i$ and

$$\mu(Q_i) \leq d^2\mu(\tilde{Q}_i) \leq \frac{d^2}{\alpha}\mu(\Omega \cap \tilde{Q}_i) \leq \frac{d^2}{\alpha}\mu(\Omega \cap Q_i),$$

showing the remaining inequality of (iii). Finally, as before, property (ii) follows immediately by the Lebesgue differentiation theorem, since given $x \in \Omega \setminus \cup_{i\in I}Q_i$, we can find a decreasing sequence of dyadic cubes $x \in R_k$ with the property that $\mu(\Omega \cap R_k) < \alpha\mu(R_k)$, and hence $\chi_\Omega(x) = 0$, a.e. $x \in \Omega \setminus \cup_{i\in I}Q_i$. □

In what follow, we shall use the following definition: given a doubling weight ω and $t > 0$, $\Omega_t \subset Q$ is the set so that $|\Omega_t| = t$,

$$\omega_Q(x) \geq \omega_Q^*(t),\ \forall x \in \Omega_t \quad \text{and} \quad \omega_Q(x) \leq \omega_Q^*(t),\ \forall x \in Q\backslash\Omega_t. \tag{6}$$

Similarly, we shall consider the set $\Omega_t^\omega \subset Q$ so that $\omega(\Omega_t^\omega) = t$,

$$\omega^{-1}(x) \geq (\omega_Q^{-1})^{*,\omega}(t),\ \forall x \in \Omega_t^\omega \quad \text{and} \quad \omega^{-1}(x) \leq (\omega_Q^{-1})^{*,\omega}(t),\ \forall x \in Q\backslash\Omega_t^\omega. \tag{7}$$

Then, by Lemma 2.2, given $0 < \alpha < 1$ and $t \leq \alpha|Q|$, there exists a family of cubes Q_i^α, $i \in I$ with disjoint interiors such that

$$|\Omega_t\backslash(\cup Q_i^\alpha)| = 0, \quad \text{and} \quad \frac{\alpha}{2^{2n}}\left|Q_i^\alpha\right| \leq \left|\Omega_t \cap Q_i^\alpha\right| < \alpha\left|Q_i^\alpha\right|, \tag{8}$$

and also, if $t \leq \alpha w(Q)$, there exists a family of cubes (we denote them in the same way) $Q_i^\alpha \subset Q, i \in I$ with disjoint interiors such that

$$\omega(\Omega_t^\omega\backslash(\cup Q_i^\alpha)) = 0, \quad \text{and} \quad \frac{\alpha}{d^2}\omega(Q_i^\alpha) \leq \omega(\Omega_t^\omega \cap Q_i^\alpha) \leq \alpha\omega(Q_i^\alpha), \tag{9}$$

where d is the constant in the doubling property of w.

Definition 2.3. *We will say that a locally integrable positive function ω in $\mathbb{R}^n$ has a finite index if there exist constants $\alpha \in (0,1), \lambda, \gamma > 0$ such that for any cube Q we have*

$$\frac{\omega_Q^*(s)}{\omega_Q^*(t)} \leq \gamma \Big(\frac{s}{t}\Big)^{-\lambda}, \text{ for any } 0 < s \leq t < \alpha\,|Q|\,. \tag{10}$$

We will denote by $\operatorname{ind}(w)$ *the infimum of all $\lambda > 0$, for which* (10) *holds.*

Remark 2.4. We shall show that if (10) holds for some $\alpha \in (0,1)$ then it holds for *any* $\alpha \in (0,1)$ maybe with different constants $\lambda, \gamma > 0$.

Lemma 2.5. *If* $\operatorname{ind}(\omega) < 1$ *then there exist constants $\alpha, \beta \in (0,1)$ such that for any subset E of the cube Q we have*

(i) $|E| < \alpha\,|Q| \quad \Longrightarrow \quad \omega(E) < \beta\omega(Q)$;
(ii) $\omega(E) \leq (1-\beta)\omega(Q) \quad \Longrightarrow \quad |E| \leq (1-\alpha)\,|Q|\,.$

Proof. Using the hypothesis and (4), with $f = w_Q^*$, there exist $\gamma \geq 1$, $0 < \lambda < 1$ and $0 < \tilde{\alpha} < 1$ so that, if $|E| \leq \tilde{\alpha}|Q|$,

$$\begin{aligned}\omega(E) = \int_E \omega(x)\,dx &\leq \int_0^{|E|} \omega_Q^*(\tau)\,d\tau \leq \gamma \Big(\frac{|E|}{\tilde{\alpha}|Q|}\Big)^{1-\lambda} \int_0^{\tilde{\alpha}|Q|} w_Q^*(\tau)\,d\tau \\ &\leq \gamma \Big(\frac{|E|}{\tilde{\alpha}|Q|}\Big)^{1-\lambda} \omega(Q),\end{aligned}$$

and hence, taking $0 < \alpha < \tilde{\alpha}$ so that $\beta := \gamma\big(\frac{\alpha}{\tilde{\alpha}}\big)^{1-\lambda} < 1$, we obtain (i). To prove (ii) note that if $\omega(E) \leq (1-\beta)\omega(Q)$ we have

$$\omega(Q \backslash E) = \omega(Q) - \omega(E) \geq \beta\omega(Q),$$

and, by (i), it follows that $|Q \backslash E| \geq \alpha\,|Q|$; i.e., $|E| \leq (1-\alpha)\,|Q|$. □

Corollary 2.6. *If* $\operatorname{ind}(\omega) < 1$, *then w is a doubling weight.*

Proof. Let $\alpha, \beta \in (0,1)$ as in Lemma 2.5 and let us consider numbers $r_0 = 1 < r_1 < r_2 < \ldots < r_K \leq 2$ such that

$$\omega(r_{i-1}Q) = (1-\beta)\omega(r_iQ), \quad i = 1, \ldots, K.$$

Then from Lemma 2.5 (ii) we have that $|r_{i-1}Q| \leq (1-\alpha)\,|r_iQ|\,, i = 1, \ldots, K$ and therefore $r_{i-1}^n \leq (1-\alpha)r_i^n$. As $r_0 = 1$, $(1-\alpha)^{-\frac{K}{n}} \leq r_K \leq 2$, and thus, $K \leq n \ln 2 (\ln \frac{1}{1-\alpha})^{-1}$. Choosing

$$d = \Big(\frac{1}{1-\beta}\Big)^{1+\left(\frac{n \ln 2}{\ln \frac{1}{1-\alpha}}\right)},$$

the result follows. □

Lemma 2.7. *If* $\operatorname{ind}(\omega) < 1$, *there exist constants $\gamma \geq 1$, $0 < \beta < 1$ and $0 < \lambda < 1$ such that*

$$\frac{(\omega_Q^{-1})^{*,\omega}(s)}{(\omega_Q^{-1})^{*,\omega}(t)} \leq \gamma \Big(\frac{s}{t}\Big)^{-\lambda}, \quad 0 < s \leq t < (1-\beta)\omega(Q). \tag{11}$$

Proof. Let $0<\beta<1$ as in Lemma 2.5 and set $0<t\le(1-\beta)\omega(Q)$. Consider the set Ω_t^ω and the family of cubes $Q_i^{(1-\beta)}$ as in (7) and (9). Observe that by Lemma 2.5 (ii), and writing for simplicity, $Q_i^{1-\beta}=Q_i$, we have $|\Omega_t\cap Q_i|\le(1-\alpha)\,|Q_i|$. Then

$$\begin{aligned}\sum_{i\in I}|Q_i| &= \sum_{i\in I}|\Omega_t\cap Q_i|+\sum_{i\in I}\int_{Q_i\cap\Omega_t^c}\omega(x)^{-1}\omega(x)\,dx\\ &\le(1-\alpha)\sum_{i\in I}|Q_i|+(\omega_Q^{-1})^{*,\omega}(t)\sum_{i\in I}\omega(Q_i),\end{aligned}$$

and consequently,

$$\begin{aligned}\sum_{i\in I}|Q_i| &\le \frac{(\omega_Q^{-1})^{*,\omega}(t)}{\alpha}\sum_{i\in I}\omega(Q_i)\le\frac{(\omega_Q^{-1})^{*,\omega}(t)}{\alpha}\frac{d^2}{1-\beta}\sum_{i\in I}\omega(\Omega_t\cap Q_i)\\ &\le\frac{d^2}{\alpha(1-\beta)}(\omega_Q^{-1})^{*,\omega}(t)t.\end{aligned}$$

Finally,

$$\int_0^t(\omega_Q^{-1})^{*,\omega}(\tau)\,d\tau\le\omega(\Omega_t)\le\sum_{i\in I}|Q_i|\le\frac{d^2}{\alpha(1-\beta)}(\omega_Q^{-1})^{*,\omega}(t)t,$$

and the result follows from Lemma 2.1 with $f=(\omega_Q^{-1})^{*,\omega}$. □

Proposition 2.8. *If $\omega\in A_p$, then the following two conditions hold:*

(i) *for every Q and every $0<t\le\alpha|Q|$ with $0<\alpha<1$,*

$$\omega_Q^*(t)\le\frac{1}{t}\int_0^t(\omega_Q)^*(\tau)\,d\tau\lesssim\|w\|_{A_p}\omega_Q^*(t);$$

(ii) $\operatorname{ind}(\omega)<1$.

Proof. Part (i) was proved in [10], but we present here a new and short proof using our technique. Let us fix $0<\alpha<1$ and let $0<t\le\alpha\,|Q|$. Let us consider the set $\Omega_t\subset Q$ as in (6) and (8). Since $w\in A_p$, and $|Q_i^\alpha\setminus\Omega_t|\ge(1-\alpha)\,|Q_i^\alpha|$, we have

$$\begin{aligned}\|w\|_{A_p} &\ge\left(\frac{1}{|Q_i^\alpha|}\int_{Q_i^\alpha}\omega_Q(x)\,dx\right)\left(\frac{1}{|Q_i^\alpha|}\int_{Q_i^\alpha\setminus\Omega_t}(\omega_Q(x))^{1-p'}\,dx\right)^{p-1}\\ &\ge\left(\frac{1}{|Q_i^\alpha|}\int_{Q_i^\alpha}\omega_Q(x)\,dx\right)\frac{1}{\omega_Q^*(t)}(1-\alpha)^{p-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^t(\omega_Q)^*(\tau)\,d\tau &=\int_{\Omega_t}\omega(x)\,dx\le\sum_{i\in I}\int_{Q_i^\alpha}\omega_Q(x)\,dx\le\frac{\|w\|_{A_p}}{(1-\alpha)^{p-1}}\omega_Q^*(t)\sum_{i\in I}|Q_i^\alpha|\\ &\le\frac{\|w\|_{A_p}2^{2n}}{\alpha(1-\alpha)^{p-1}}\omega_Q^*(t)\sum_{i\in I}|\Omega_t\cap Q_i^\alpha|\le\frac{\|w\|_{A_p}2^{2n}}{\alpha(1-\alpha)^{p-1}}\omega_Q^*(t)t,\end{aligned}$$

and (i) is proved. Finally, (ii) follows by Lemma 2.1. □

Remark 2.9. Observe that if $w \in A_p$, we have proved that (10) holds, for every $0 < \alpha < 1$.

We are now ready to prove our main result.

Theorem 2.10. *The following conditions are equivalent:*

(i) $\omega \in A_\infty$.
(ii) $\operatorname{ind}(\omega) < 1$.

Proof. By Proposition 2.8, it remains to prove that if $\operatorname{ind}(\omega) < 1$, there exists $p \geq 1$ so that $\omega \in A_p$. Let β as in Lemma 2.7 and let $r > 1$. Then,

$$\begin{aligned} A_Q &:= \left(\frac{1}{\omega(Q)} \int_Q (\omega(x))^{-r} \omega(x)\, dx \right)^{\frac{1}{r}} = \left(\frac{1}{\omega(Q)} \int_0^{\omega(Q)} ((\omega_Q^{-1})^{*,\omega}(\tau))^r\, dx \right)^{\frac{1}{r}} \\ &\leq \left(\frac{1}{(1-\beta)\omega(Q)} \int_0^{(1-\beta)\omega(Q)} ((\omega_Q^{-1})^{*,\omega}(\tau))^r\, d\tau \right)^{\frac{1}{r}}. \end{aligned}$$

By (11) , if we take r so that $r\lambda < 1$, we obtain that

$$((\omega_Q^{-1})^{*,\omega}(\tau))^r \leq \gamma \left(\frac{\tau}{(1-\beta)\omega(Q)} \right)^{-r\lambda} ((\omega_Q^{-1})^{*,\omega}((1-\beta)\omega(Q)))^r$$

and by simple integration we obtain that

$$\begin{aligned} A_Q &= \frac{1}{(1-r\lambda)^{1/r}} ((\omega_Q^{-1})^{*,\omega}((1-\beta)\omega(Q))) \\ &\leq \frac{1}{(1-r\lambda)^{1/r}} \frac{1}{(1-\beta)\omega(Q)} \int_0^{\omega(Q)} ((\omega_Q^{-1})^{*,\omega}(\tau) d\tau \\ &\leq \frac{1}{(1-r\lambda)^{1/r}} \frac{1}{(1-\beta)\omega(Q)} \int_Q \frac{\omega(x)}{\omega(x)} dx = \frac{1}{(1-r\lambda)^{1/r}} \frac{|Q|}{(1-\beta)\omega(Q)}, \end{aligned}$$

from which it follows that $w \in A_{r'}$. □

Corollary 2.11. *The following conditions are equivalent:*

(i) $w \in A_\infty$.
(ii) *For every* $0 < \alpha < 1$, *there exists* $C_\alpha > 0$ *so that, for every* Q *and every* $0 < t \leq \alpha|Q|$,
$$\omega_Q^*(t) \leq \frac{1}{t} \int_0^t \omega_Q^*(s)\, ds \leq C_\alpha \omega_Q^*(t).$$
(iii) *There exists* $0 < \alpha < 1$ *and* $C_\alpha > 0$ *so that, for every* Q *and every* $0 < t \leq \alpha|Q|$,
$$\omega_Q^*(t) \leq \frac{1}{t} \int_0^t \omega_Q^*(s)\, ds \leq C_\alpha \omega_Q^*(t).$$

Corollary 2.12. *If* φ *is a nonnegative concave function in* $(0, \infty)$, *then*

$$w \in A_\infty \implies \varphi(w) \in A_\infty.$$

Proof. This is an immediate consequence of Theorem 2.10 and the fact that every concave function on $(0,\infty)$ is increasing. ☐

From Theorem 2.10, we can define the following constants for every A_∞ weight: given $0<\alpha<1$,

$$\|\omega\|_{A_\infty,\alpha}=\sup_Q \sup_{0<t\le\alpha|Q|}\frac{\frac{1}{t}\int_0^t \omega_Q^*(s)\,ds}{\omega_Q^*(t)}.$$

We observe that if $0<\alpha_1<\alpha_2<1$, then $1\le\|\omega\|_{A_\infty,\alpha_1}\le\|\omega\|_{A_\infty,\alpha_2}$ and hence we can define the least constant by

$$\|\omega\|_{A_\infty,0}=\lim_{\alpha\to 0}\|\omega\|_{A_\infty,\alpha}\ge 1.$$

In [4] (see also [12]) the following characterization of A_∞ was given (see also condition P7 in [3]): $\omega\in A_\infty$ if and only if, there exists $c>0$ so that, for every Q,

$$\int_Q M(\omega\chi_Q)(x)\,dx\le c\omega(Q),$$

where M is the Hardy-Littlewood maximal operator and

$$\|\omega\|_{A_\infty,M}=\sup_Q\frac{\int_Q M(\omega\chi_Q)(x)\,dx}{\omega(Q)}$$

is called the A_∞ Fujii constant. Using the n-dimensional Riesz inequality for the maximal operator [2],

$$(Mf)^*(t)\le\frac{K}{t}\int_0^t f^*(s)\,ds, \tag{12}$$

we obtain that, for $0<\alpha<1$,

$$\int_Q M(\omega\chi_Q)(x)\,dx\le\int_0^{|Q|}\left(M(\omega\chi_Q)\right)^*(t)\,dt\le\frac{1}{\alpha}\int_0^{\alpha|Q|}\left(M(\omega\chi_Q)\right)^*(t)\,dt\le$$
$$\frac{K}{\alpha}\int_0^{\alpha|Q|}\frac{1}{t}\int_0^t\omega_Q^*(s)\,ds\,dt\le\frac{K\|\omega\|_{A_\infty,\alpha}}{\alpha}\int_0^{\alpha|Q|}\omega_Q^*(t)\,dt\le\frac{K\|\omega\|_{A_\infty,\alpha}}{\alpha}\omega(Q),$$

and hence,

$$\|\omega\|_{A_\infty,M}\le K\inf_{0<\alpha<1}\frac{\|\omega\|_{A_\infty,\alpha}}{\alpha},$$

with K the constant in (12). Using Theorem 2.10, we can extend this result as follows:

Corollary 2.13. *If T is a sublinear operator of joint weak-type $(1,1;\infty,\infty)$ (see [2]) then, for every $w\in A_\infty$,*

$$\|\omega\|_{A_\infty,T}:=\sup_Q\frac{1}{w(Q)}\int_Q|T(w\chi_Q)(x)|\,dx<\infty.$$

Moreover,

$$\|\omega\|_{A_\infty,T}\lesssim\inf_{0<\alpha<1}\frac{\|\omega\|_{A_\infty,\alpha}}{\alpha}+1.$$

Proof. By hypothesis, we have that

$$(Tf)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)\,ds + \int_t^\infty f^*(s)\,\frac{ds}{s},$$

and hence

$$(T(w_Q))^*(t) \lesssim \frac{1}{t}\int_0^t w_Q^*(s)\,ds + \int_t^{|Q|} w_Q^*(s)\,\frac{ds}{s}.$$

Thus, using Theorem 2.10, we obtain that

$$\begin{aligned}\int_Q |T(w\chi_Q)(x)|\,dx &\leq \int_0^{|Q|} (T(w_Q))^*(t)\,dt \\ &\leq \frac{\|\omega\|_{A_\infty,\alpha}}{\alpha}\int_0^{\alpha|Q|} w_Q^*(s)\,ds + \int_0^{|Q|}\int_t^{|Q|} w_Q^*(s)\,\frac{ds}{s}\,dt \\ &\lesssim \left(\frac{\|\omega\|_{A_\infty,\alpha}}{\alpha}+1\right) w(Q),\end{aligned}$$

and the result follows. □

In particular, $\|w\|_{A_\infty,T}$ is finite for T the Hilbert transform or any singular integral operator (see also [1]).

It is known that $\omega \in A_p$ if and only if $\omega \in A_\infty$ and $\omega^{1-p'} \in A_\infty$. This can be proved using our characterization in the following easy way:

Corollary 2.14. *A weight ω belongs to A_p, $1 < p < \infty$ if and only if* $\operatorname{ind}(\omega) < 1$ *and* $\operatorname{ind}(\omega^{-1}) < p - 1$.

Proof. If $\omega \in A_p$, by Proposition 2.8 we have that $\operatorname{ind}(\omega) < 1$, and since $\sigma(x) = \omega(x)^{1-p'} \in A_{p'}$ we also have that $\operatorname{ind}(\omega(x)^{1-p'}) < 1$ and hence $\operatorname{ind}(w^{-1}) < p - 1$.

Conversely, if $\operatorname{ind}(\omega) < 1$, we have by Remark 2.9 and (10) with $\alpha = 1/2$, that there exists $\gamma > 0$ so that

$$\int_Q \omega(x)\,dx = \int_0^{|Q|} \omega_Q^*(s)\,ds \leq 2\int_0^{\frac{|Q|}{2}} \omega_Q^*(s)\,ds \leq \ \gamma\omega_Q^*\Big(\frac{|Q|}{2}\Big)\,|Q|\,.$$

Similarly, if $\operatorname{ind}(\omega^{-1}) < p - 1$ we have that

$$\int_Q (\omega(x)^{1-p'}\,dx \leq \gamma'(w_Q^{1-p'})^*\Big(\frac{|Q|}{2}\Big)\,|Q|\,,$$

and the result follows by observing that $(\omega_Q^{1-p'})^*(|Q|/2) = \Big(\omega_Q^*(|Q|/2)\Big)^{1-p'}$. □

3. Reverse Hölder inequality and the $(p-\varepsilon)$-condition

It is well-known that if $w \in A_p$, then there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$. This is the so-called $(p-\varepsilon)$-condition. Also, there exists $r > 1$ so that

$$\left(\frac{1}{|Q|}\int_Q \omega(x)^r\,dx\right)^{1/r} \leq \frac{C}{|Q|}\int_Q \omega(x)\,dx.$$

This condition is referred as the reverse Hölder inequality, $w \in RH_r$, and the least constant satisfying this inequality is denoted by $\|w\|_{RH_r}$.

Our first result in this section establishes a relation between $\operatorname{ind}(\omega)$ and the values of r so that $w \in RH_r$. This is of the same nature as [8, Theorem 1.1] (see also some related arguments in [11, 6, 7]), but we point out that our proof follows very easily using our main result Theorem 2.10.

Theorem 3.1. *If $w \in A_\infty$, then w satisfies the reverse Hölder's inequality for every $1 < r < (\operatorname{ind}(\omega))^{-1}$.*

Proof. Since $\operatorname{ind}(\omega) < 1$, we have that, if $r\lambda < 1$ with λ as in (10),

$$\begin{aligned}\int_Q \omega(x)^r\,dx &\le \int_0^{|Q|} \omega_Q^*(s)^r\,ds \le \frac{1}{\alpha}\int_0^{\alpha|Q|} \omega_Q^*(s)^r\,ds \\ &\lesssim \frac{1}{\alpha}\omega_Q^*(\alpha|Q|)^r(\alpha|Q|)^{\lambda r}\int_0^{\alpha|Q|}\frac{1}{s^{\lambda r}}ds = \frac{\omega_Q^*(\alpha|Q|)^r|Q|}{1-\lambda r},\end{aligned}$$

and therefore

$$\left(\frac{1}{|Q|}\int_Q \omega(x)^r\,dx\right)^{1/r} \lesssim \frac{\omega_Q^*(\alpha|Q|)}{(1-\lambda r)^{1/r}} \le \frac{1}{\alpha}\frac{w(Q)}{(1-\lambda r)^{1/r}},$$

and the result follows. □

Our next result establishes a relation between the norm $\|w\|_{A_\infty,\alpha}$ and the values of r so that $w \in RH_r$.

Theorem 3.2. *If $w \in A_\infty$ and $\varphi_\alpha(x) = (\frac{\alpha(1+x)}{x})^{\frac{1}{1+x}}$, then w satisfies the reverse Hölder's inequality for every $1 < r < \sup_{0<\alpha<1}\varphi_\alpha^{-1}(\|w\|_{A_\infty,\alpha}) + 1$.*

Proof. Let us write $r = 1+\varepsilon$, with $\varepsilon < \sup_{0<\alpha<1}\varphi_\alpha^{-1}(\|w\|_{A_\infty,\alpha})$ and let α be so that $\varepsilon < \varphi_\alpha^{-1}(\|w\|_{A_\infty,\alpha})$. Since φ_α is decreasing, we have that $\|w\|_{A_\infty,\alpha} < \varphi_\alpha(\varepsilon)$. Now, for every Q,

$$\begin{aligned}\int_Q \omega(x)^{1+\varepsilon}\,dx &\le \int_0^{|Q|}\omega_Q^*(t)^{1+\varepsilon}\,dt \le \frac{1}{\alpha}\int_0^{\alpha|Q|}\omega_Q^*(t)^{1+\varepsilon}\,dt \\ &\le \frac{1}{\alpha}\int_0^{\alpha|Q|}\omega_Q^*(t)\bigg(\int_0^t(\omega_Q^*(s)\,ds\bigg)^{\varepsilon} t^{-\varepsilon}\,dt \\ &\le \frac{1}{1+\varepsilon}\left(\frac{\omega(Q)}{\alpha|Q|}\right)^{1+\varepsilon}|Q| + \frac{\varepsilon}{\alpha(1+\varepsilon)}\int_0^{\alpha|Q|}\left(\frac{1}{t}\int_0^t \omega_Q^*(s)\,ds\right)^{1+\varepsilon}dt \\ &\le \frac{1}{1+\varepsilon}\left(\frac{\omega(Q)}{\alpha|Q|}\right)^{1+\varepsilon}|Q| + \frac{\varepsilon}{\alpha(1+\varepsilon)}\|\omega\|_{A_\infty,\alpha}^{1+\varepsilon}\int_0^{\alpha|Q|}\omega_Q^*(t)^{1+\varepsilon}\,dt \\ &\le \frac{1}{1+\varepsilon}\left(\frac{\omega(Q)}{\alpha|Q|}\right)^{1+\varepsilon}|Q| + \frac{\varepsilon}{\alpha(1+\varepsilon)}\|\omega\|_{A_\infty,\alpha}^{1+\varepsilon}\int_Q \omega(x)^{1+\varepsilon}\,dx.\end{aligned}$$

Since $\frac{\varepsilon}{\alpha(1+\varepsilon)}\|\omega\|_{A_\infty,\alpha}^{1+\varepsilon} < 1$, we obtain that

$$\frac{1}{|Q|}\int_Q \omega(x)^{1+\varepsilon}\,dx \leq \frac{1}{1-\frac{\varepsilon}{\alpha(1+\varepsilon)}\|\omega\|_{A_\infty,\alpha}^{1+\varepsilon}}\frac{1}{(1+\varepsilon)\alpha^{1+\varepsilon}}\left(\frac{\omega(Q)}{|Q|}\right)^{1+\varepsilon},$$

and the result follows. □

The following theorem was proved in [9] and gives a clear relation of the optimal ε in the $(p-\varepsilon)$-condition with the optimal exponent r in the reverse Hölder condition for the so-called dual weight $\sigma = w^{1-p'}$. We present a new proof keeping track of the constant.

Theorem 3.3. *Given $w \in A_p$ and $1 < q < p$,*

$$w \in A_q \quad\Longleftrightarrow\quad \sigma \in RH_{\frac{p-1}{q-1}}.$$

Moreover,

$$\|w\|_{A_q} \leq \|\sigma\|_{RH_{\frac{p-1}{q-1}}}^{p-1}\|w\|_{A_p}$$

and

$$\|\sigma\|_{RH_{\frac{p-1}{q-1}}} \leq 2(\|w^{1-q'}\|_{A_\infty,\frac{1}{2}})^{\frac{q-1}{p-1}}.$$

Proof. The first inequality is immediate and already known [9], since if $\sigma \in RH_{\frac{p-1}{q-1}}$

$$\begin{aligned}\|w\|_{A_q} &= \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \sigma(x)^{\frac{p-1}{q-1}}\,dx\right)^{q-1} \\ &\leq \|\sigma\|_{RH_{\frac{p-1}{q-1}}}^{p-1}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \sigma(x)\,dx\right)^{(p-1)} \leq \|\sigma\|_{RH_{\frac{p-1}{q-1}}}^{p-1}\|w\|_{A_p}.\end{aligned}$$

Now let us assume that $w \in A_q$. Then, by Theorem 2.8 applied to $w^{1-q'} \in A_\infty$, we have that, for every $0 < t \leq |Q|/2$,

$$t\left(w_Q^{1-q'}\right)^*(t) \leq \int_0^t \left(w_Q^{1-q'}\right)^*(s)\,ds \leq \|w^{1-q'}\|_{A_{\infty,\frac{1}{2}}}\, t\left(w_Q^{1-q'}\right)^*(t).$$

Using this inequality for $t = |Q|/2$ we will obtain the reverse Hölder inequality we are looking for:

$$\left(\frac{1}{|Q|}\int_Q \sigma(x)^{\frac{p-1}{q-1}}\,dx\right)^{\frac{q-1}{p-1}} = \frac{1}{|Q|^{\frac{q-1}{p-1}}}\left(\int_Q w(x)^{1-q'}\,dx\right)^{\frac{q-1}{p-1}}$$

$$= \frac{1}{|Q|^{\frac{q-1}{p-1}}}\left(\int_0^{|Q|}\left(w_Q^{1-q'}\right)^*(s)\,ds\right)^{\frac{q-1}{p-1}} \leq \left(\frac{2}{|Q|}\right)^{\frac{q-1}{p-1}}\left(\int_0^{|Q|/2}\left(w_Q^{1-q'}\right)^*(s)\,ds\right)^{\frac{q-1}{p-1}}$$

$$\leq \|w^{1-q'}\|_{A_\infty,\frac{1}{2}}^{\frac{q-1}{p-1}}\left(\left(w_Q^{1-q'}\right)^*\left(\frac{|Q|}{2}\right)\right)^{\frac{q-1}{p-1}} = \|w^{-\frac{1}{q-1}}\|_{A_\infty,\frac{1}{2}}^{\frac{q-1}{p-1}}\left(w_Q^{1-p'}\right)^*\left(\frac{|Q|}{2}\right)$$

$$\leq \|w^{1-q'}\|_{A_\infty,\frac{1}{2}}^{\frac{q-1}{p-1}}\frac{2}{|Q|}\int_0^{|Q|/2}\left(w_Q^{1-p'}\right)^*(s)\,ds \leq 2\|w^{1-q'}\|_{A_\infty,\frac{1}{2}}^{\frac{q-1}{p-1}}\left(\frac{1}{|Q|}\int_Q w^{1-p'}(x)\,dx\right)$$

$$= 2\|w^{1-q'}\|_{A_\infty,\frac{1}{2}}^{\frac{q-1}{p-1}}\left(\frac{1}{|Q|}\int_Q \sigma(x)\,dx\right).$$

□

References

[1] E. Agora, M.J. Carro, and J. Soria, *Characterization of the weak-type boundedness of the Hilbert transform on weighted Lorentz spaces,* J. Fourier Anal. Appl. **19** (2013), no. 4, 712–730.

[2] C. Bennett and R. Sharpley, *Interpolation of Operators,* Academic Press, Boston, 1988.

[3] J. Duoandikoetxea, F.J. Martín-Reyes, and S. Ombrosi, *On the A_∞ conditions for general bases,* Math. Z. **282** (2016), 955–972.

[4] N. Fujii, *Weighted bounded mean oscillation and singular integrals*, Math. Japon **22** (1977/1978), 529–534.

[5] J. García-Cuerva and J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, Mathematical Studies **116**, North-Holland, 1985.

[6] T. Hytönen and C. Pérez, *Sharp weighted bounds involving A_∞*, Analysis and PDE **6** (2013), 777–818.

[7] T. Hytönen, C. Pérez, and E. Rela, *Sharp reverse Hölder property for A_∞ weights on spaces of homogeneous type*, J. Funct. Anal. **263** (2012), no. 12, 3883–3899.

[8] R. Johnson and C.J. Neugebauer, *Homeomorphisms preserving A_p*, Rev. Mat. Iberoamericana **3** (1987), no. 2, 249–273.

[9] R. Johnson and C.J. Neugebauer, *Change of variable results for A_p- and reverse Hölder* RH_r-*classes*, Trans. Amer. Math. Soc. **328** (1991), no. 2, 639–666.

[10] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[11] C.J. Neugebauer, *The precise range of indices for the RH_r- and A_p- weight classes*, preprint available at https://arxiv.org/pdf/math/9809162.pdf.

[12] J.M. Wilson, *Weighted inequalities for the dyadic square function without dyadic A_∞*, Duke Math. J. **55** (1987), 19–49.

(I. Asekritova) Department of Mathematics, Linköping University, 581 83 Linköping, Sweden.
Email address: irina.asekritova@liu.se

(M.J. Carro) Department of Mathematics and Computer Science, University of Barcelona, Gran Via 585, E-08007 Barcelona, Spain.
Email address: carro@ub.edu

(N. Kruglyak) Department of Mathematics, Linköping University, 581 83 Linköping, Sweden.
Email address: natan.kruglyak@liu.se

(J. Soria) Department of Mathematics and Computer Science, University of Barcelona, Gran Via 585, E-08007 Barcelona, Spain.
Email address: soria@ub.edu